Department of Mathematics and Statistics
University of New Mexico

Homework 4 **Math 510: Real Analysis 1** **Fall 2020**

Instructions: Complete 6 problems to get full credit (60 points), the seventh problem is bonus for extra 10 points.
This homework is due on Thursday Sept 17, 2020.

1. (Rudin Chapter 3 #3) If $s_1 = \sqrt{2}$, and $s_{n+1} = \sqrt{2+\sqrt{s_n}}$ $(n = 1, 2, 3, \dots)$, prove that $\{s_n\}_{n=1}^{\infty}$ converges, and that $s_n < 2$ for $n = 1, 2, 3, \dots$.

2. (Rudin Chapter 3 #5) For any two real sequences $\{a_n\}$, $\{b_n\}$, prove that

$$\limsup_{n\to\infty}(a_n + b_n) \le \limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n,$$

provided the sum on the right is not of the form $\infty - \infty$.

3. (Rudin Chapter 3 #7, Qual Jan 2001 #5) Assume that the series $\sum_{n=1}^{\infty} a_n$ converges and $a_n \ge 0$. Prove that $\sum_{n=1}^{\infty} \frac{\sqrt{a_n}}{n}$ converges.

4. (Rudin Chapter 3 #23) Suppose $\{p_n\}$ and $\{q_n\}$ are Cauchy sequences in a metric space X. Show that the sequence $d(p_n, q_n)$ converges. (There is a hint in the book.)

5. (a) (Qual Aug 2004 #4) Let $s_n := 1 + \dfrac{1}{2} + \cdots + \dfrac{1}{n}$. Prove that $s_{2n} - s_n \ge \dfrac{1}{2}$, and use it to show that the series $1 + \frac{1}{2} + \cdots + \frac{1}{n} + \dots$ diverges.

(b) (Qual Aug 2005 #5) Consider the following series $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \dots$. In other words, the general term is $-1/n$ if $n = 2^k$ for some $k = 1, 2, \dots$, and its equal to $1/n$ otherwise. Show that the series diverges.

6. (Qual Fall 2008 #1, Rudin Chapter 3 #14(a)(b)) Let $\{s_n\}_{n\ge1}$ be a sequence of real numbers. Consider the sequence of its arithmetic means, defined to be for each $n \ge 1$,

$$\sigma_n = \frac{s_1 + s_2 + \cdots + s_n}{n}.$$

Show that if the sequence s_n converges to s, then the sequence $\{\sigma_n\}_{n\ge1}$ also converges and to the same limit s. Does convergence of the sequence of averages σ_n imply convergence of the given sequence $\{s_n\}_{n\ge1}$? Explain.

7. (Rudin Chapter 3 #8, compare to Qual Aug 1998 #2) If $\sum_{n=0}^{\infty} a_n$ is convergent and if $\{b_n\}_{n=0}^{\infty}$ is monotonic and bounded prove that $\sum_{n=0}^{\infty} a_n b_n$ converges.

Practice Problems on your own

These are all problems from Rudin Chapter 3.

#1. Prove that convergence of $\{s_n\}$ implies convergence of $\{|s_n|\}$.

#2. Calculate $\lim_{n\to\infty}(\sqrt{n^2+n}-n)$.

#4. Find the upper and lower limits of the sequence $\{s_n\}$ defined by

$$s_1 = 0; \quad s_{2m} = \frac{s_{2m-1}}{2}; \quad s_{2m+1} = \frac{1}{2} + s_{2m}.$$

#9. Find the radius of convergence of each of the following power series:

$$\text{(a)} \sum_{n=0}^{\infty} n^3 z^n, \quad \text{(b)} \sum_{n=0}^{\infty} \frac{2^n}{n!} z^n, \quad \text{(c)} \sum_{n=1}^{\infty} \frac{2^n}{n^2} z^n, \quad \text{(d)} \sum_{n=0}^{\infty} \frac{n^3}{3^n} z^n.$$

#10. Suppose that the coefficients of the power series $\sum_{n=0}^{\infty} a_n z^n$ are integers, infinitely many of which are distinct from zero. Prove that the radius of convergence is at most 1.

#20. Suppose $\{p_n\}$ is a Cauchy sequence in a metric space X, and some subsequence $\{p_{n_k}\}$ converges to a point $p \in X$. Prove that the full sequence $\{p_n\}$ converges to p.