MATH 510 - Introduction to Analysis I - Fall 2020
Homework # 5 (continuous functions)

Do 4 problems for full credit (40 points) and one additional problem for 10 Bonus points. Homework to be turned in on Tuesday October 8, 2020 at 11:59pm.

1. (Rudin Chapter 4 #18) Every rational x can be written in the form $x = m/n$, where $n > 0$, and m and n are integers without any common divisors. When $x = 0$, we take $n = 1$. Consider the function f defined on $\mathbb{R}$ by $f(x) = 0$ when x is an irrational number, and $f(x) = 1/n$ when $x = m/n$. Prove that f is continuous at every irrational point, and that f has a simple discontinuity at every rational point.

2. (Qual Jan 2002 #2, Jan 2003 #3) Let E, F be metric spaces, and let f be a function defined on E into F.

(a) Give the "$\epsilon - \delta$" of continuity.

(b) Show that $f : E \to F$ is continuous at c if and only if given any sequence $\{x_n\}$ in E convergent to c, then $f(x_n)$ converges to $f(c)$ when $n \to \infty$.

(c) Suppose $f^{-1}(V)$ is open in E for every open V in F. Show that f is continuous on E.

3. Suppose f is a continuous mapping from a metric space X to a metric space Y.

(a) **(Qual Aug 1997 #2, Aug 2015 #1)** Prove that if K is a compact subset of X, then $f(K)$ is a compact subset of Y. Discuss a consequence that could be stated (but not proved) in a first year calculus text.

(b) **(Qual Aug 2004 #1(b))** If $G \in X$ is an open set in X is it necessarilly true that $f(G)$ is open? Give an example.

(c) **(Qual Jan 2018 #2)** Prove that if K is a connected compact subset of X, then $f(K)$ is a connected compact subset of Y.

4. (Qual Jan 2009 #4) Let (X, d) be a compact metric space and F a mapping $F : X \to X$ such that

$$d\big(F(x), F(y)\big) < d(x, y) \quad \text{for all} \quad x, y \in X.$$

(a) Show that the function $g : X \to [0, \infty)$, defined by $g(x) = d\big(x, F(x)\big)$, is a continuous function whose minimum must be zero.

(b) Show that the equation $F(x) = x$ has exactly one solution, i.e., F has exactly one fixed point.

5. (Qual August 2010 #1) Let $f : \mathbb{R} \to \mathbb{R}$, assume $f(1) > 0$, f is continuous at $x = 0$, and that for all $x, y \in \mathbb{R}$

$$f(x + y) = f(x)f(y).$$

Show that there exists a real number $a \in \mathbb{R}$ such that $f(x) = e^{ax}$. Assume known all the properties of the exponential and logarithmic functions that you may need, in particular may use the fact that they are continuous functions on their domain.

On your own

1. (Rudin Chapter 4 #1) Suppose f is a real function defined on $\mathbb{R}$ which satisfies

$$\lim_{h\to 0}[f(x+h)-f(x-h)]=0, \qquad \text{for all } x\in\mathbb{R}.$$

Does this imply that f is continuous?

2. (Rudin Chapter 4 #3) Let f be a continuous real function on a metric space X. Let $Z(f)$ (the *zero set* of f) be the set of all $p \in X$ at which $f(p)=0$. Prove that $Z(f)$ is closed.

3. (Rudin Chapter 4 #4) Let f and g be continuous mappings of a metric space X into a metric space Y, and let E be a dense subset of X. Prove that $f(E)$ is dense in $f(X)$. If $g(p)=f(p)$ for all $p\in E$, prove that $g(p)=f(p)$ for all $p\in X$. (In other words, a continuous mapping is det ermined by its values on a dense subset of its domain.)

4. (Rudin Chapter 4 #7) If $E\subset X$ and if f is a function de fined on X, the *restriction* of f to E is the function g whose do main of definition is E, such that $g(p)=f(p)$ for all $p\in E$. Define f a nd g on $\mathbb{R}^2$ by: $f(0,0)=g(0,0)=0$, and $f(x,y)=xy^2/(x^2+y^4)$, $g(x,y)=xy^2/(x^2+y^6)$ if $(x,y)\neq(0,0)$. Prove that f is bounded on $\mathbb{R}^2$, that g is unbounded in every neighborhood of $(0,0)$, and that f is not continuous at $(0,0)$; nevertheless, the restrictions of both f and g to every straight line in $\mathbb{R}^2$ are continuous!

5. (Qual Aug 2003 # 2) Let V be a vector space over the reals.

(a) Define a norm $\|\cdot\|: V\to\mathbb{R}$ on V, and show it is continuous on V.

(b) Let C^0 be the vector space of real valued continuous functions on $\mathbb{R}$, define $\|\cdot\|_a : C^0\to\mathbb{R}$ for each $f\in C^0$ by

$$\|f\|_a=\sup_{x\in\mathbb{R}}(1+|x|^a)|f(x)|, \quad \text{where} \quad a>0.$$

Show that for each $a>0$, $\|\cdot\|_a$ is a norm, and show that $(C^0,\|\cdot\|_a)$ is a complete metric space.

6. (Qual Jan 2006 # 2, see also Qual Aug 2013 #1) Let $C([0,1])$ be the metric space of continous real-valued functions on $[0,1]$, with the uniform metric (i.e. $d(f,g)=\sup_{x\in[0,1]}|f(x)-g(x)|$). Denote by B the closed unit ball in $C([0,1])$, that is,

$$B=\{f\in C([0,1]) : \|f\|_\infty = d(f,0)=\sup_{x\in[0,1]}|f(x)|\leq 1\}.$$

Show that B is not compact. **Hint:** Consider the sequence of functions $x, x^2, x^3, \dots$.

7. (Qual Jan 2017 #2) Let (X,d) be a metric space. A function $f: X\to\mathbb{R}$ is said to be *lower semicontinuous at a point* $x_0\in X$ if for any sequence $x_n\to x_0$

$$\liminf_{n\to\infty} f(x_n)\geq f(x_0).$$

A function is said to be *lower semicontinuous on* X if it is lower semicontinuous at every point. Prove that is f is lower semicontinuous on X, then $\{x\in X : f(x)>a\}$ is open for any $a\in\mathbb{R}$. **Hint:** *it may be helpful to prove the contrapositive.*