MATH 510 - Introduction to Analysis I - Fall 2020
Homework # 6 (more on continuous functions, uniform continuity)

Choose 5 problems for 50 points, and you can do the 6th problem for 10 bonus points. The homework is due on Thursday October 15, 2020 at 11:59pm.

1. (Qual Jan 2019 #2) Supose $f : (a, b) \to \mathbb{R}$ is a uniformly continuous function on a bounded open interval $(a, b) \subset \mathbb{R}$.

(a) **(compare to Rudin Chapter 4 #11)** Prove that if $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence in (a, b), then its image under f, $\{f(x_n)\}_{n=1}^{\infty}$, is also a Cauchy sequence.

(b) **(compare to Rudin Chapter 4 #8)** Prove that f is a bounded function.

2. (Rudin Chapter 4 #10 - Continuous functions on a compact set are uniformly continuous) Complete the details of the following alternative proof of Theorem 4.19: If f is not uniformly continuous, then for some $\epsilon > 0$ there are sequences $\{p_n\}$ and $\{q_n\}$ in X such that $d_X(p_n, q_n) \to 0$ but $d_Y(f(p_n), f(q_n)) > \epsilon$. Use theorem 2.37 to obtain a contradiction.

3. (Rudin Chapter 4 #14 - Fixed point on $\mathbb{R}$) Let $I = [0, 1]$ be the closed unit interval. Suppose that f is a continuous mapping of I into I. Prove that $f(x) = x$ for at least one $x \in I$.

4. (Qual Aug 1999 #6) Let S be a bounded open interval $S = (a, b)$. Let $\overline{S} = [a, b]$ be its closure. Let f be a function defined on S.

(b) Show that if f is uniformly continuous on S then it can be extended continuously to $\overline{S}$.

(c) Give an example of a continuous function on S that cannot be extended continuously to $\overline{S}$.

5. (Qual Aug 2002 #4) The space $C([a, b])$ of real-valued continuous functions over the interval $[a, b]$ is a complete metric space with the distance induced by the sup norm,

$$d(f, g) = \sup_{x \in [a,b]} |f(x) - g(x)|.$$

(a) Given a real-valued function F defined on $C([a, b])$, define continuity and uniform continuity of F.

(b) Let $F : C([a, b]) \to \mathbb{R}$ be given by $F(f) = \int_a^b f(t)\, dt$. Show that F is uniformly continuous. (Operate here with integrals freely.)

6. (Qual Jan 1999 #3) Let $f : [a, b] \to \mathbb{R}$, where $[a, b]$ is a compact interval.

(a) Define what it means that f is uniformly continuous.

(b) Prove that if f is continuous, then it is uniformly continuous.

(c) Show that the graph of f, $G_f := \{(x, y) : x \in [a, b], y = f(x)\}$, can be covered with a finite number of rectangles such that the area of their union is smaller than ϵ for any given $\epsilon > 0$. (**Remark:** This implies that G_f has measure zero in $\mathbb{R}^2$.)

On your own

7. (Qual Fall 2009 #2) [Assume known that the sine function is a continuous function]

(a) Show that the function $f(x) = \sin\left(\frac{\pi}{x}\right)$ is continuous on the interval $(0,1)$.

(b) Is f uniformly continuous on $(0,1)$?

(c) For a real-valued function g defined on a metric space (X,d) let

$$w(r) = \sup\{|g(x) - g(x')| : d(x,x') \le r\}.$$

Show that g is a uniformly continuous function if and only if $\lim_{r\to 0} w(r) = 0$.

8. (Qual Aug 2008 #2) Let K be a compact subset of $\mathbb{R}$, and let f be a real-valued function defined on K. Denote by Γ_f the graph of f, a subset of $\mathbb{R}^2$, more precisely,

$$\Gamma_f = \{(x,y) \in \mathbb{R}^2 : x \in K, y = f(x)\}.$$

Show that f is continuous on K if and only if its graph Γ_f is a compact subset of $\mathbb{R}^2$.

9. (Qual Jan 2005 #1) Let K be a compact metric space with metric d and let f be a continuous real-valued function defined on K. Prove that the graph of the function f

$$\Gamma_f = \{(x,y); x \in K, y = f(x)\}$$

is a compact set in the metric space $(K \times \mathbb{R}, \rho)$, where

$$\rho\big((x_1,y_1),(x_2,y_2)\big) = d(x_1,x_2) + |y_1 - y_2|.$$

10. (Qual Jan 2006 #3) Let (X,d), and (Y,ρ) be metric spaces, $f : X \to Y$ a continuous function. Prove that if X is compact and f is one-to-one and onto (bijective) then $f^{-1} : Y \to X$ is continuous. Will the statement remain true if X is not compact? Explain.

11. (Qual Aug 2008 #3, Rudin Chapter 4 #20) Let (X,d) be a metric space. Let E be a non-empty subset of X. Define the distance from $x \in X$ to E by

$$\rho_E(x) = \inf_{y\in E} d(x,y).$$

(a) Prove that $\rho_E(x) = 0$ if and only if x belongs to the closure of E, denoted $\overline{E}$.

(b) Prove that ρ_E is a uniformly continuous function on X.
(Hint: show that $|\rho_E(x_1) - \rho_E(x_2)| \le d(x_1,x_2)$.

12. (Qual Fall 2009 #4)

(a) The projection map $p : \mathbb{R}^2 \to \mathbb{R}$ is given by $p(x, y) = x$. Determine if the projection map is (i) a continuous map, (ii) an open map, or (iii) a closed map. We are using the standard metrics in the corresponding Euclidean spaces.

(b) Let $f : X \to Y$ be a continuous map between the metric spaces (X, d) ad (Y, ρ). If $K \subset X$ is compact, is it true that $f(K)$ is compact? Explain.

13. (Qual Aug 2016 #2(b), Qual Aug 2000 #5(b)) Let (X, ρ) be a compact metric space, and let $f : X \to X$ be an isometry, i.e., $\rho\big(f(x), f(y)\big) = \rho(x, y)$ for every $x, y \in X$. Show that f is bijective (that is a one-to-one and onto function).

14. (Qual Aug 2017 #5) Let A be a symmetric (real-valued) matrix and $\lambda(\xi) = A\xi \cdot \xi/|\xi|^2$, for $\xi \in \mathbb{R}^n \setminus \{0\}$, where $|\xi|^2 = \sum_{j=1}^n x_j^2$ for $\xi = (x_1, x_2, \ldots, x_n)$.

(a) Show that λ achieves its infimum on $\mathbb{R}^n \setminus \{0\}$. Hint: use that λ is constant on every ray emanating from the origin.

(b) Let $\xi_0 \in \mathbb{R}^n \setminus \{0\}$ be such that $\lambda_0 = \lambda(\xi_0) = \inf\{\lambda(\xi) : \xi \in \mathbb{R}^n \setminus \{0\}$. Show that ξ_0 is an eigenvector of A with eigenvalue λ_0, $A\xi_0 = \lambda_0\xi_0$. Hint: for any $\xi \in \mathbb{R}^n \setminus \{0\}$ consider the function $\phi(t) = \lambda(\xi_0 + t\xi)$ for $t > 0$.