MATH 510 - Introduction to Analysis I - Fall 2020
Homework # 7 (differentiable functions)

Choose 4 problems out of the first 5 for 40 points and the fifth problem is for 10 additional bonus points. This homework is due on Tuesday October 27, 2020 at 11:59pm.

1. (Qual Aug 1998 #1) Let f be a real-valued function defined on a closed and bounded interval $[a, b]$ in $\mathbb{R}$.

(a) State the Mean Value Theorem for such functions f.

Assuming that f satisfies the conditions necessary for the Mean Value Theorem to hold, prove or disprove the following two statements:

(b) If $f'(x) > 0$ on (a, b) then f is strictly increasing on $[a, b]$.

(c) If f is strictly increasing on $[a, b]$, then $f'(x) > 0$ on (a, b).

2. (Qual Aug 2019 #2)

(a) Let $f : [0, 1] \to \mathbb{R}$ be a continuous function that is differentiable at some $x_0 \in (0, 1)$. Prove that there exists $L > 0$ such that $|f(x) - f(x_0)| \leq L|x - x_0|$ for all $x \in [0, 1]$.

(b) Consider the function $f(x) = x^{3/2} \sin(1/x)$ if $0 < x \leq 1$ and $f(x) = 0$ when $x = 0$. Prove that for every $L > 0$ there exists $x, y \in [0, 1]$ such that $|f(x) - f(y)| > L|x - y|$. (*Hint:* Analyze the derivative f'.)

3. (Qual Aug 1997 #4) State Taylor's Theorem with remainder in one variable. Use the Mean Value Theorem to prove Taylor's Theorem.

4. (Qual Jan 2018 #3) Let $f : [0, \infty) \to (0, \infty)$ be differentiable and increasing and $g : [0, \infty) \to (0, \infty)$ differentiable and decreasing functions. Define $F(x) = \int_0^x f(t)\,dt$ and $G(x) = \int_0^x g(t)dt$.

(a) Prove that $\lim_{x\to\infty} F(x) = \infty$.

(b) Prove that if $\lim_{x\to\infty} f(x)$ and $\lim_{x\to\infty} g(x)$ are finite and positive, then $\lim_{x\to\infty} F(x)/G(x)$ is finite.

5. Rudin Chapter 5, Exercise #13.

On your own

6. (Qual Aug 1997 #3, Qual Jan 2019 #6, Rudin Ch. 5 #1) Let f be a real function on $\mathbb{R}$. Suppose there exists some $p > 1$ such that $|f(x) - f(y)| \leq |x - y|^p$ for all $x, y \in \mathbb{R}$. Prove that f is a constant.

7. (Qual Aug 2000 #3) Show that if $f : (a, b) \to \mathbb{R}$ is real analytic in its domain and f is not identically equal to zero, then the zeros of f are isolated; i.e., if $f(c) = 0$ for some $c \in (a, b)$, there exists a $\delta > 0$ such that $f(x) \neq 0$ for all $x \in (a, b)$ satisfying $0 < |x - c| < \delta$. Remember that a function is real analytic at a point x_0 if it is infinitely differentiable at the point x_0 and if the Taylor series centered at x_0 has a positive radius of convergence.

8. (Qual Jan 2001 #7, Qual Aug 2014 #3 -two derivatives-, Qual Jan 2020 #1 -one derivative-)

(a) Suppose $f(x)$ is defined on an open interval containing x, and $f(x)$ is three times differentiable on this interval. Show that,

$$f'''(x) = \lim_{h\to 0} \frac{1}{h^3} \left(f(x + h) - 3f(x) + 3f(x - h) - f(x - 2h)\right).$$

(b) Give an example when this limit exists but $f'''(x)$ does not.

9. (Qual Aug 2002 #2) A real-valued function on an open interval in $\mathbb{R}$ is called *convex* if no point on the line segment between any two points of its graph lies below the graph. Show that if a function is differentiable and convex, then no point of the graph lies below any point of any tangent line to the graph. State these conditions in precise analytical terms and prove them.

10. (Qual Jan 2004 #3) State the Mean Value Theorem for a function f on a closed bounded interval $[a, b]$. Use it to prove the following statement: if f is a three times continuously differentiable function defined on $\mathbb{R}$ and there exists points $x_1 < x_2 < x_3 < x_4$ such that $f(x_1) = f(x_2) = f(x_3) = f(x_4)$, then there exists a point $\xi \in (x_1, x_4)$ such that $f^{(3)}(\xi) = 0$.

11. (Qual Aug 2004 #2) Let P be a polynomial of degree n in $\mathbb{R}$. Suppose that all the roots of P are real and distinct. Use the Mean Value Theorem to prove that for $k = 1, \ldots, n-1$, all the roots of $P^{(k)}$ are real and distinct.

12. (Qual Jan 2005 #4) Suppose that f is differentiable in the closed interval $[a, b]$ and that its second derivative f'' exists in the open interval (a, b). Suppose also that

$$f(a) = f(b), \quad f'(a) = f'(b) = 0.$$

Show that there exists two points $c_1, c_2 \in (a, b)$, $c_1 \neq c_2$ such that

$$f''(c_1) = f''(c_2).$$

13. (Qual Aug 2005 #2, Homework 2 #1)

(a) Let $f : [a, b] \to \mathbb{R}$ be twice differentiable. Assume $f''(x) < 0$ for all $a \leq x \leq b$. Show that for all $0 \leq t \leq 1$,

$$f(a)t + f(b)(1-t) \leq f\big(at + b(1-t)\big).$$

Explain the geometric meaning of the above inequality.

(b) Use part **(a)** to show that if $a, b \geq 0$, then the following inequality holds for all $0 \leq t \leq 1$,

$$a^t b^{1-t} \leq at + b(1-t).$$

14. (Qual Jan 2006 #2, Qual Jan 2013 #3)

(a) Define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} e^{-1/x^2} & \text{if } x > 0 \\ 0 & \text{if } x \leq 0. \end{cases}$$

Prove that f has derivatives of all orders, with $f^{(n)}(0) = 0$ for all n.

(b) Given interval (a, b), construct a positive function $g : \mathbb{R} \to \mathbb{R}$ such that g is infinitely differentiable on $\mathbb{R}$, and is zero if and only if x is not on (a, b).

15. (Qual Aug 2007 #2) Let $f : \mathbb{R} \to \mathbb{R}$ be a twice differentiable function. Given real numbers x and y, prove that there exists a number ξ in between x and y such that

$$f(y) = f(x) + f'(x)(y-x) + \frac{1}{2} f''(\xi)(y-x)^2.$$

16. (Qual Aug 2007 #3, Rudin Ch. 5 #15) Let $f : \mathbb{R} \to \mathbb{R}$ be a twice differentiable function. Assume that for all $x \in \mathbb{R}$, $|f(x)| \leq A$ and $|f''(x)| \leq B$. Prove, using Taylor's Theorem as in Problem 15, that $|f'(x)| \leq 2\sqrt{AB}$.

17. (Qual Aug 2008 #4, Rudin Ch. 5 #3) Let g be a differentiable function, $g : \mathbb{R} \to \mathbb{R}$. Assume that g has a bounded derivative. Fix $\epsilon > 0$, and let

$$f(x) = x + \epsilon g(x).$$

Show that for ϵ small enough, the function f is injective (or one-to-one).

18. Rudin Chapter 5 Exercise #2.

19. (Qual Aug 2010 #2) Assume f is twice differentiable on $[a, b]$ with continuous second derivative on $[a, b]$, and that it has value zero at the endpoints, $f(a) = f(b) = 0$.

(a) Define $g : [a, b] \to \mathbb{R}$ by $g(x) = \frac{f(x)}{x-a}$ if $a < x \leq b$ and $f(x) = c$ when $x = a$. Find c so that g is continuous on $[a, b]$. With that value of c, show that g is continuously differentiable on $[a, b]$.

(b) Prove that there exists a constant $M > 0$ such that for all $x \in [a, b]$ the following inequality holds $|f(x)| \leq M|x - a|\,|x - b|$.

20. (Qual Aug 2013 #1) Suppose that f is a real-valued differentiable function on $[a, b]$ such that f' exists and is continuous on $[a, b]$. Given $\epsilon > 0$ prove that there exists a $\delta > 0$ such that

$$\left| \frac{f(t) - f(x)}{t - x} - f'(x) \right| < \epsilon$$

whenever $t, x \in [a, b]$ are any two points satisfying $0 < |t - x| < \delta$ (in other words, f is in some sense "uniformly differentiable"). Hint: use MVT.

19. (Qual Jan 2018 #3) Let $f : [0, \infty) \to (0, \infty)$ be differentiable and increasing and $g : [0, \infty) \to (0, \infty)$ differentiable and decreasing functions. Define $F(x) = \int_0^x f(t)\, dt$ and $G(x) = \int_0^x g(t) dt$.

(a) Prove that $\lim_{x\to\infty} F(x) = \infty$.

(b) Prove that if $\lim_{x\to\infty} f(x)$ and $\lim_{x\to\infty} g(x)$ are finite and positive, then $\lim_{x\to\infty} F(x)/G(x)$ is finite.

20. (Qual Aug 13 #2)

(a) Let $f : [0, 1] \to \mathbb{R}$ be a continuous function that is differentiable at some $x_0 \in (0, 1)$. Prove that there exists $L > 0$ such that $|f(x) - f(x_0)| \leq L|x - x_0|$ for all $x \in [0, 1]$.

(b) Consider the function $f(x) = x^{3/2} \sin(1/x)$ if $0 < x \leq 1$ and $f(x) = 0$ when $x = 0$. Prove that for every $L > 0$ there exists $x, y \in [0, 1]$ such that $|f(x) - f(y)| > L|x - y|$. (*Hint:* Analyze the derivative f'.)