MATH 510 - Introduction to Analysis I - Fall 2020
Homework # 8 (Integration)

Homework due on Tuesday Nov 10th, 2020 at 11:59pm Choose 5 problems to turn in for 50 points, the sixth you can do to get 10 bonus points.

1. (Rudin Chapter 6 #2) Suppose $f \geq 0$, f is continuous on $[a, b]$, and $\int_a^b f(x)\,dx = 0$. Prove that $f(x) = 0$ for all $x \in [a, b]$.

2. (Rudin Chapter 6 #7) Suppose f is a real function on $(0, 1]$ and $f \in \mathcal{R}$ on $[c, 1]$ for every $c > 0$. Define $\int_0^1 f(x)\,dx = \lim_{c\to 0^+} \int_c^1 f(x)\,dx$ if this limit exists (and is finite).

(a) If $f \in \mathcal{R}$ on $[0, 1]$, show that this definition of the integral agrees with the old one.

(b) Construct a function f such that the above limit exists, although it fails to exist with $|f|$ in place of f.

3. (Qual Spring 2005 #3) Let $f \in C[0, 1]$ and suppose $f(t) > 0$ for all $t \in [0, 1]$. Define $\theta_n > 0$ by the following equation:

$$\int_0^{\theta_n} f(x)\,dx = \frac{1}{n}\int_0^1 f(x)\,dx.$$

find the following limit $\lim\limits_{n\to\infty} n\theta_n$.

4. (Qual Spring 2005 #2) Let f, g be real-valued continuous functions defined on the interval $[0, 1]$, i.e. $f, g \in C[0, 1]$. Consider the uniform metric on $C[0, 1]$ given by $\rho(f, g) := \sup_{t\in[0,1]} |f(t) - g(t)|$. For $f \in C[0, 1]$ define $F(f)$ as the continuous function defined for each $t \in [0, 1]$ by $F(f)(t) = \displaystyle\int_0^t uf(u)\,du$. Show that $F : C[0, 1] \to C[0, 1]$ is a contraction, i.e.

$$\rho(F(f), F(g)) \leq \alpha\rho(g, f), \quad \forall f, g \in C[0, 1].$$

with some $\alpha \in (0, 1)$. Explain why this implies that the equation

$$f(t) = \int_0^t uf(u)\,du, \quad t \in [0, 1]$$

has a unique solution $f \in C[0, 1]$.

5. (Qual Jan 2014 #4) Suppose $f : [a, b] \to \mathbb{R}$ is Riemann integrable. Show that f^2 is also a Riemann integrable function by proving that for any $\epsilon > 0$ there exists a partition P such that $U(f^2, P) - L(f^2, P) < \epsilon$. You may not apply the theorem which states that the composition of a continuous function with an integrable function is integrable.

6. (Qual Jan 2020 #4, Qual Aug 2014 #5) Prove that the function $f(x) = \sin(1/(x - 1))$ if $x \neq 1$ and $f(1) = 0$ is Riemann integrable on $[0, 2]$.

On your own

7. Exercises in Rudin:

(i) **Chapter 6 # 6** (Cantor function is Riemann integrable),

(ii) **Chapter 6 # 10** (Holder's inequality),

(iii) **Chapter 6 # 11** (triangle inequality),

(iv) **Chapter 6 # 15** (uncertainty principle),

(v) **Chapter 6 # 17** (integration by parts),

(vi) **Chapter 6 # 18** (rectifiable curves).

8. (Qual Jan 2002 #5) Let $a, b \in \mathbb{R}$, $a < b$ and let f be a real-valued function on $[a, b]$.

(a) Define the Riemann integral of f between a and b.

(b) Show that f is Riemann integrable on $[a, b]$ if and only if, for each $\epsilon > 0$, there exists step functions f_1, f_2 on $[a, b]$ such that $f_1(x) \leq f(x) \leq f_2(x)$ for each $x \in [a, b]$, and

$$\int_a^b [f_2(x) - f_1(x)]\, dx < \epsilon.$$

9. (Qual Jan 2003 #6) Prove the following: Let f be a Riemann integrable function on $[a, b]$. For $a \leq x \leq b$, put $F(x) = \displaystyle\int_a^x f(t)\, dt$. Then F is continuous on $[a, b]$; furthermore, if f is continuous at a point x_0 of $[a, b]$, then F is differentiable at x_0, and $F'(x_0) = f(x_0)$.

10. (Qual Jan 2004 #5) Given a Riemann integrable function f on $[0, 1]$, prove the convergence of the following series and find its sum:

$$\sum_{k=1}^{\infty} \frac{1}{2^k} \int_1^2 f\left(\frac{x}{2^k}\right)\, dx.$$

11. (Qual Jan 2004 #6) Let f be a continuously differenctiable function with $f'(0) \neq 0$. For $x > 0$, let $\xi = \xi(x)$ be a number in $[0, x]$ such that

$$\int_0^x f(u)\, du = f(\xi)x.$$

Prove the following limit exists and find it: $\displaystyle\lim_{x \to 0+} \frac{\xi(x)}{x}$.

12. (Qual Jan 2002 #4)

(i) Prove the Contraction Mapping Principle: Given a complete metric space (M, ρ), let $T : M \to M$ be a map such that there exists a constant $0 < c < 1$ such that $\rho(Tu, Tv) \leq c\rho(u, v)$ for all $u, v, \in M$. Then there exists a unique $w \in M$ such that $Tw = w$ (that is w is a unique fixed point).

(ii) Let $T : C([0,1]) \to C([0,1])$ be defined by $Tf(x) = x + \displaystyle\int_0^x tf(t)\,dt$. Prove that T satisfies the hypothesis of the contractive mapping principle. Show that the fixed point is a solution to the differential equation $f'(x) = xf(x) + 1$.

Note: Can assume known that $C([0,1])$ is a complete metric space (Problem # 3(b) in the same qualifying).

13. (Qual Spring 2006 #4) Let $[a,b]$ and $[c,d]$ be closed intervals in $\mathbb{R}$, and let $f(x,y)$ be a continuous real-valued function on $\{(x,y) \in \mathbb{R}^2 : x \in [a,b], y \in [c,d]\}$. Show that the function $g : [c,d] \to \mathbb{R}$, defined by

$$g(y) = \int_a^b f(x,y)\,dx, \quad \forall y \in [c,d],$$

is continuous. First you need to explain why the function g is well-defined.

14. (Qual Spring 2007 #4) Let $f : [0,\infty) \to \mathbb{R}$ be a decreasing function. Assume that $\left|\int_0^\infty f(x)\,dx\right| < \infty$. Show that

$$\lim_{x\to\infty} xf(x) = 0.$$

15. Hint: Verify the statements first for polynomials.

(a) **(Qual Spring 2007 #5)** Let $f : [0,1] \to \mathbb{R}$ be a continuous function. Show that

$$\lim_{n\to\infty} n \int_0^1 x^n f(x)\,dx = f(1).$$

(b) **(Qual Spring 2009 #1)** Let $f \in \mathcal{C}([0,1])$. Show that $\displaystyle\lim_{n\to\infty} \int_0^1 x^n f(x)dx = 0$.

(c) **(Qual Fall 2009 #1)** Let $f \in \mathcal{C}([0,1])$. Show that $\displaystyle\lim_{n\to\infty} \frac{\int_0^1 x^n f(x)dx}{\int_0^1 x^n dx} = f(1)$.

16. (Qual Aug 2010 #3)

(a) Let (X, d) be a metric space, and $F : X \to \mathbb{R}$. Define what it means for F to be uniformy continuous on X.

(b) Suppose $f : [a, b] \to \mathbb{R}$ is Riemann integrable, define for $x \in [a, b]$, $F(x) := \int_a^x f(t)\,dt.$. Show that F is uniformly continuous.

(c) Let (K, d) be a compact metric space, and let $F : K \to \mathbb{R}$ be a continuous function. Show that F is uniformly continuous.

17. (Qual Aug 2012 #4) Let $f : [a, b] \to \mathbb{R}$ be a function which is bounded and Riemann integrable on $[c, b]$ for every $c \in (a, b)$. Prove that f is Riemann integrable on all of $[a, b]$.

18. The *integral test* for series says: Let $f : [1, \infty) \to \mathbb{R}$ be a monotone decreasing non-negative function. Then the sum $\sum_{n=1}^{\infty} f(n)$ is convergent if and only if the improper integral $\int_1^{\infty} f(x)\,dx := sup_{N>0} \int_1^N f(x)\,dx$ is finite.

(a) **Qual Aug 2012 #2, Rudin Chapter 6 #8)** Prove the integral test.

(b) **(Qual Jan 2013 #4)** Show by constructing counterexamples that if the hypothesis of monotone decreasing non-negative function is replaced by continuous non-negative function on intervals $[1, N]$ for all $N > 0$ then both directions of the if and only if above are false.

19. (Qual Aug 2013 #4) Suppose f is Riemann integrable on $[0, A]$ for all $0 < A < \infty$, that $\lim_{x\to\infty} f(x) = 1$, and $t > 0$. Prove that

$$\lim_{t\to 0^+} \int_0^{\infty} te^{-tx} f(x)\,dx = 1.$$

20. (Qual Aug 2015 #3) Let $f : [0, 1] \to \mathbb{R}$ be a continuous function that is also differentiable on $[0, 1]$ and such that $M := \sup_{t\in[0,1]} |f'(t)| < \infty$. Prove that for each $n \in \mathbb{N}$,

$$\left| \sum_{j=0}^{n-1} \frac{f(j/n)}{n} - \int_0^1 f(t)\,dt \right| \le \frac{M}{2n}.$$

21. (Qual Jan 2017 #3) Show that if $f : [a, b] \to \mathbb{R}$ is a continuos function then f is Riemann integrable.

22. (Qual Aug 2019 #4)

(a) Prove that for all $n \in \mathbb{N}$, the function $\ln(e^x + n^{-1})/(1 + x^2)^2$ is Riemann integrable on $[0, \infty)$.

(b) Evaluate $\displaystyle\lim_{n\to\infty} \int_0^{\infty} \frac{\ln(e^x + n^{-1})}{(1 + x^2)^2}\,dx.$