MATH 510 - Introduction to Analysis I - Fall 2020
Homework #9 (uniform convergence)

The following problems appeared in Qualifying exams in the past. Do all 5 problems for a total of 50 points. Homework due on Tuesday Dec 1st, 2020 at 11:59pm.

1. (Qual Jan 2001 #6) Consider the sequences of functions $f_n(x)$ and $g_n(x)$ on the interval $[0,1]$,

$$f_n(x) = \begin{cases} 1 & x < 1/n \\ 0 & \text{otherwise} \end{cases}, \qquad g_n(x) = \begin{cases} x & x < 1/n \\ 0 & \text{otherwise} \end{cases}.$$

Do these sequences converge uniformly on $[0,1]$? If yes, find the limit function.

2. (Qual Aug 2014) Suppose $f_n, g_n : \mathbb{R} \to \mathbb{R}$ are two sequences of functions converging uniformly on $\mathbb{R}$ to functions f, g respectively.

(a) Show that if both sequences are uniformly bounded (there is $M > 0$ such that $|f_n(x)| \leq M$ for all $n \geq 1$ and for all $x \in \mathbb{R}$, and similarly for g_n), then the product $f_n g_n$ converges uniformly to fg.

(b) Show by example that the conclusion in part (a) may fail to hold if the sequences are not assumed to be uniformly bounded.

3. (Qual Aug 2016 #5) Suppose $f : [a,b] \to \mathbb{R}$ is continuously differentiable on its domain. Given any $\epsilon > 0$, prove that there exists a polynomial $P(x)$ such that

$$|f(x) - P(x)| < \epsilon, \qquad \text{and} \qquad |f'(x) - P'(x)| < \epsilon$$

for every $x \in [a,b]$.

4. (Qual Jan 2018 #5) Let $\{\phi_n\}_{n=1}^{\infty}$ be a sequence of non-negative Riemann integrable functions on $[-1,1]$ satisfying

(i) $\int_{-1}^{1} \phi_n(t)\, dt = 1$ for all $n \geq 1$.

(ii) For every $\delta > 0$ the sequence ϕ_n converges uniformly to zero on $[-1,-\delta] \cup [\delta, 1]$.

Prove that if $f : [-1,1] \to \mathbb{R}$ is Riemann integrable and continuous at $x = 0$, then

$$\lim_{n\to\infty} \int_{-1}^{1} f(t)\phi_n(t)\, dt = f(0).$$

5. (Qual Jan 2007 #6) Given a power series $\sum_{n=0}^{\infty} a_n x^n$, let $\alpha = \limsup_{n\to\infty} |a_n|^{1/n}$. Let $R = 1/\alpha$ if $0 < \alpha < \infty$, $R = \infty$ if $\alpha = 0$, and $R = 0$ if $\alpha = \infty$.

(a) Show that if $R > 0$ then the series converges absolutely whenever $|x| < R$ to a function that we will denote $f(x)$.

(b) Show that if $0 < K < R$ then the power series converges uniformly to $f(x)$ on $[-K, K]$.

(c) Show that if $R > 0$, then the series can be differentiated term by term, and the differentiated series converges to $f'(x)$ for $|x| < R$.

ON YOUR OWN

6. (Qual Aug 2001 #5) Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of non-negative continuous functions on the interval $[0,1]$, such that the sequence is bounded at every point. Prove that this sequence is uniformly bounded on some interval $D \subset [0,1]$.

7. (Qual Jan 2003 #7 and Aug 2011 #3) Prove the following.
If $\{f_n\}_{n\in\mathbb{N}}$ is a sequence of continuous functions on a set E in a metric space, and if $f_n \to f$ uniformly on E, then f is continuous on E. As part of your argument give the precise definition of uniform convergence.

8. (Qual Aug 2003 #5) Let $\sum_{n=1}^{\infty} u_n(x)$ converge to $F(x)$ for each $x \in [a,b]$. Let $u_n'(x)$ exist and be continuous on $[a,b]$ and let $\sum_{n=1}^{\infty} u_n'(x)$ converge uniformly to $g(x)$ on $[a,b]$. Show that F is differentiable and $F' = g$. State all theorems you use. **Hint:** Fundamental Theorem(s) of Calculus may be helpful.

9. (Qual Jan 2004 #4) Suppose that for a sequence of real valued functions $\{f_n\}$ defined on the interval $[0,1]$

$$\sum_{n=1}^{\infty} \sup_{x\in[0,1]} |f_{n+1}(x) - f_n(x)| < \infty.$$

Prove that $\{f_n\}$ converges uniformly on $[0,1]$.

10. (Qual Aug 2004 #4) Suppose that f is uniformly continuous in $\mathbb{R}$ and K is a continuous function in $\mathbb{R}$ such that $K \geq 0$, $K = 0$ outside $[-1,1]$, and $\int_{-1}^{1} K(x)\,dx = 1$. Define

$$f_n(x) := n \int_{\mathbb{R}} K\big(n(x-y)\big) f(y)\,dy.$$

Prove that $\{f_n\}$ converges uniformly to f in the real line.

11. (Qual Jan 2005 #3) Let $\{a_n\}_{n\geq 1}$ be a sequence of positive real numbers such that the series $\sum_{n=0}^{\infty} a_n$ is convergent.

(a) Show that the series $\sum_{n=0}^{\infty} a_n x^n$, is absolutely convergent for $|x| \leq 1$. Define the function $f : [-1,1] \to \mathbb{R}$ by the power series,

$$f(x) := \sum_{n=0}^{\infty} a_n x^n.$$

Show that f is differentiable for $|x| < 1$. Find an explicit formula for the derivative of f in terms of the data sequence $\{a_n\}$.

(b) Show that the function f defined on part **(a)** is continuous at $x = 1$.

Show that it is not necessarily true that f is differentiable at $x = 1$.

12. (Qual Jan 2006 #1) Let $\{r_k\}_{k\geq 1}$ be an enumeration of the rationals in $(0,1)$, and let $f_k(x) = H(x - r_k)$ where $H(x)$ is the Heavyside function: $H(x) = 0$ if $x < 0$, and $H(x) = 1$ if $x \geq 0$.

(a) Show that $f(x) = \displaystyle\sum_{k=1}^{\infty} \frac{f_k(x)}{2^k}$ is uniformly convergent on $x \in [0,1]$.

(b) Show that f is strictly increasing on $[0,1]$, with $f(0) = 0$ and $f(1) = 1$.

(c) Show that $\displaystyle\int_0^1 f(x)\,dx = 1 - \sum_{k=1}^{\infty} \frac{r_k}{2^k}$. Justify your reasoning.

13. (Qual Fall 2007 #4) Assume E is a compact subset of $\mathbb{R}^n$ and $f_n : E \to \mathbb{R}$ is a sequence of continuous functions satisfying: (i) $f_1(x) \geq f_2(x) \geq f_3(x) \geq \dots$, that is the sequence is decreasing, and (ii) there is a continuous function $f : E \to \mathbb{R}$ such that $\lim_{n\to\infty} f_n(x) = f(x)$ for all $x \in E$.

(a) Prove that f_n converges to f uniformly on E.

(b) Prove that (a) is false if the assumption (i) is removed.

14. (Qual Fall 2008 #5) Define for each positive integer n the function $h_n : \mathbb{R} \to \mathbb{R}$ by,

$$h_n(x) = \begin{cases} 1 & \text{if } x \in [0, 2^{n-1}), \\ -1 & \text{if } x \in [2^{n-1}, 2^n), \\ 0 & \text{otherwise.} \end{cases}$$

(a) Verify that the series $\sum_{n=1}^{\infty} \frac{h_n(x)}{2^n}$ converges uniformly on $\mathbb{R}$.

(b) Verify that the series defined on part (a) converges pointwise on $\mathbb{R}$ to $\chi_{[0,1)}(x) = \begin{cases} 1 & \text{if } x \in [0,1), \\ 0 & \text{otherwise.} \end{cases}$

(c) Denote by f_N the function given by the following partial sums, $f_N(x) = \sum_{n=1}^{N} \frac{h_n(x)}{2^n}$.
Is it true that $\lim_{N\to\infty} \int_{\mathbb{R}} f_N(x)\,dx = \int_{\mathbb{R}} \lim_{N\to\infty} f_N(x)\,dx$?

Under what circumstances uniform convergence of a sequence of real-valued functions guarantees that one can interchange the limit and the integral?

15. (Qual Jan 2009 #2) Let $|a| < 1$ and $f_n(t) = \sin\left((2n+1)\frac{\pi}{2}t\right)$, $t \in \mathbb{R}$.

(a) Show that for any a and t as above, the series $\displaystyle\sum_{n=1}^{\infty} a^{2n+1} f_n(t)$ converges absolutely.

(b) Determine if $f(t) = \displaystyle\sum_{n=1}^{\infty} a^{2n+1} f_n(t)$ is a continuous function of t.

(c) Show that $\displaystyle\frac{1}{\pi} \ln \frac{1+a}{1-a} - \frac{2}{\pi} a = \int_0^1 f(t)\,dt.$

16. (Qual Aug 2010 #4) Prove that the following two series converge uniformly, however one converges absolutely always and the other never.

(a) Show that $\sum_{n=1}^{\infty} \frac{\cos(nx)}{2^n}$ converges uniformly and absolutely for all $x \in \mathbb{R}$.

(b) Show that $\sum_{n=1}^{\infty} (-1)^n \left(\frac{x^n+n}{n^2}\right)$ converges uniformly for all $x \in [0,1]$, but not absolutely.

17. (Qual Jan 2012 #3) Suppose $f : \mathbb{R} \to \mathbb{R}$ is a uniformly continuous function on all of $\mathbb{R}$. Let $\{y_n\}_{n=1}^{\infty}$ be a sequence of real numbers. For each $n \in \mathbb{N}$ define a new function, $f_n(x) = f(x + y_n)$, for all $x \in \mathbb{R}$. If $\lim_{n\to\infty} y_n = 0$ show that the sequence of functions $\{f_n\}_{n=1}^{\infty}$ converges uniformly on $\mathbb{R}$.

18. (Qual Jan 2012 #4) A real valued function f on $[0,1]$ is said to be *Hölder continuous of order* α if there is a positive constant C such that $|f(x) - f(y)| \leq C|x-y|^{\alpha}$ for all $x, y \in [0,1]$. For these functions define

$$\|f\|_{\alpha} := \max_{0\leq x\leq 1} |f(x)| + \sup_{0\leq x,y\leq 1, x\neq y} \frac{|f(x)-f(y)|}{|x-y|^{\alpha}}.$$

Suppose that $0 < \alpha \leq 1$ and $\{f_n\}_{n=1}^{\infty}$ is a sequence of Hölder continuous functions of order α satisfying $\|f_n\|_{\alpha} \leq 1$ for all $n \geq 1$. Show that $\{f_n\}_{n=1}^{\infty}$ is an equicontinuous sequence. Conclude that there exists a subsequence $\{f_{n_k}\}_{k=1}^{\infty}$ which converges uniformly on $[0,1]$.

19. (Qual Aug 2013 #3) Suppose $f_k : [a, b] \to \mathbb{R}$ is a sequence of Riemann integrable functions on $[a, b]$ such that the series $\sum_{k=1}^{\infty} f_k$ is uniformly convergent.

(a) Show that $\sum_{k=1}^{\infty} f_k$ is Riemann integrable.

(b) Show that moreover, $\displaystyle\sum_{k=1}^{\infty} \int_a^b f_k(x)\,dx = \int_a^b \sum_{k=1}^{\infty} f_k(x)\,dx.$

20. (Qual Jan 2015 #5) Prove that if a power series $\sum_{k=1}^{\infty} a_k x^k$ converges for some $x_0 \neq 0$, then it converges uniformly on every interval $[-R, R]$ and the sum of the series represents a continuous function on $[-R, R]$, where $0 < R < |x_0|$.

21. (Qual Jan 2017 #4) Recall the summation by parts formula valid for any pair of real-valued sequences $\{a_n\}_{n\geq 0}$, $\{b_n\}_{n\geq 0}$, and any pair of natural numbers $p.q$ with $p \leq q$, namely

$$\sum_{n=p}^{q} a_n b_n = \sum_{n=p}^{q} (a_n - a_{n+1}) \sum_{k=p}^{n} b_k + a_{q+1} \sum_{k=p}^{q} b_k.$$

Let $f_n : [-1, 0] \to [0, \infty)$ for each $n \in \mathbb{N}$ be a sequence of functions uniformly convergent to zero on its domain. Suppose for each fixed $x \in [-1, 0]$ the sequence of non-negative numbers $\{f_n(x)\}_{n\geq 0}$ is monotone decreasing. Show that the series $\sum_{n=0}^{\infty} f_n(x) x^n$ converges uniformly on $[-1, 0]$.

22. (Qual Aug 2017 #2) Let $\mathbb{Q} \cap (0, 1) = \{x_1, x_2, \dots\}$ be the set of rational numbers in $(0, 1)$ listed as a sequence. For $n \in \mathbb{N}$ let $f_n(x) = 0$ if $0 \leq x < x_n$ and $f_n(x) = 1/3^n$ if $x_n \leq x \leq 1$.

(a) Determine if the series $\sum_{n=1}^{\infty} f_n(x)$ converges uniformly on $[0, 1]$.

(b) Show that f is an increasing function on $[0, 1]$ which is continuous at every irrational number.

(c) Show that f is discontinuous at every rational number.

23. (Qual Aug 2017 #3) Let $\{p_n\}_{n=1}^{\infty}$ be a sequence of polynomials of degree at most 100 defined on $[0, 1]$ and such that for all $n, k \in \mathbb{N}$ we have $|p_n^{(k)}(0)/k!| \leq 1$.

(a) Show that $\{p_n\}_{n=1}^{\infty}$ has a uniform convergent subsequence.

(b) Show that the limit function is a polynomial.

24. (Qual Jan 2019 #5) Let $\sum_{n=0}^{\infty} a_n x^n$ be a power series centered at 0. Suppose the series converges at some $x_0 \in \mathbb{R} \setminus \{0\}$. Let $\epsilon \in (0, |x_0|)$. *Without appealing to the radius of convergence*, prove that the series converges uniformly on any closed interval $[-|x_0| + \epsilon, |x_0| + \epsilon]$.
This result is part of the theory behind the radius of convergence, hence a solution independent of this is asked for here. Instead use the comparison test and/or Weiertrass M-test.

25. (Qual Aug 2019 #3)

(a) State the ϵ-N definition of uniform convergence of a sequence of functions $\{f_n\}_{n=1}^{\infty}$ to a function f on $\mathbb{R}$.

(b) Consider the sequence of functions given by $f_n(x) = \sqrt{x^2 + \frac{1}{n^2}}$, $\quad x \in \mathbb{R}$, $\quad n \in \mathbb{N}$. Prove that $\{f_n\}_{n=1}^{\infty}$ converges uniformly on $\mathbb{R}$.

(c) State what it means by the ϵ-N definition that the sequence of functions $\{g_n\}_{n=1}^{\infty}$ does not converge to a function g uniformly on $\mathbb{R}$.

(d) Prove that the sequence of derivatives $\{f_n'\}_{n=1}^{\infty}$ of the functions defined in part (b) does not converge uniformly on $\mathbb{R}$.

26. (Qual Jan 2020 #5) Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of real-valued functions that are continuous on $[0, 1]$ and converges to f uniformly on $[0, 1]$. Prove that

$$\lim_{n\to\infty} \int_0^{1-\frac{1}{n}} f_n(x)\,dx = \int_0^1 f(x)\,dx.$$