MATH 510 - Introduction to Analysis I
Homework # 2:

The following problems appeared in Qualifying exams in the past.

1. (Qual Aug 1997 # 1) Let X be an infinite set. For $p \in X$ and $q \in X$ define

$$d(p,q) = \begin{cases} 1 & \text{if } p \neq q \\ 0 & \text{if } p = q \end{cases}$$

Prove that this is a metric. Which subsets of the resulting metric are open? Which are closed? Which are compact?

2. (Qual Aug 1998 # 4) Determine which of the following sets (with the usual metric) are compact. If any of them is not compact, find the smallest compact set (if it exists) containing the given set.

(a) $\{1/k : \quad k \in \mathbb{N}\} \bigcup \{0\}$.

(b) $\{(x,y) \in \mathbb{R}^2 : \quad y = \sin(1/x) \quad \text{for some} \quad x \in (0,1)\}$.

(c) $\{(x,y) \in \mathbb{R}^2 : \quad |xy| \leq 1\}$.

3. (Qual Aug 1999 # 1)

(a) Define what it means that a subset S of $\mathbb{R}$ is connected.

(b) Show that the set $S = [0,1] \bigcap \mathbb{Q}$ is not connected.

(c) Show that the interval $[0,1]$ is connected in $\mathbb{R}$

4. (Qual Jan 2003 # 1) Let E be an infinite subset of a compact set K (every open cover has a finite subcover) in a metric space X. Define metric space and limit point and show that E has a limit point in K.

5. (Qual Aug 2004 # 2) Let (X,d) be a compact metric space and $\{F_n\}$ be a sequence of closed subsets of X. If $\bigcap_{n=1}^{\infty} F_n = \emptyset$, then there exists $N \geq 1$ such that $\bigcap_{n=1}^{N} F_n = \emptyset$.

6. (Qual Aug 2005 # 1) Consider $\mathbb{R}$ with the Euclidean metric.

(a) Let A be an open subset of $\mathbb{R}$. Show that A can be written as a union of a countable number of disjoint open intervals.

(b) Let B be a closed subset of $\mathbb{R}$. Construct a function $f : \mathbb{R} \to \mathbb{R}$, continuous and such that $f(x) = 0$ if and only if $x \in B$.

(c) (Not in the qual) Can you construct a function $f : \mathbb{R} \to \mathbb{R}$ that is infinitely differentiable, positive, bounded by one, and such that $f(x) = 0$ if and only if $x \in B$?

This homework is due on Wednesday Sept 14, 2005.