Harmonic Analysis: from Fourier to Haar

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APPENDIX B

Vector spaces, norms, inner products

Here we record the definitions and the main properties of vector spaces, inner-product vector spaces, normed spaces, Banach and Hilbert spaces.

We pay particular attention to the geometry of inner-product vector spaces, and state the basic theorems related to orthogonality: the Cauchy–Schwarz inequality, the Pythagorean theorem and the triangle inequality. We also discuss orthonormal bases and orthogonal projections onto closed subspaces.

B.1. Vector spaces

We begin with the definition of a vector space.

DEFINITION B.1. A *vector space over* $\mathbb{C}$ is a set V, together with operations of addition and scalar multiplication, such that if x, y, $z \in V$ and λ, λ_1, $\lambda_2 \in \mathbb{C}$ then

(i) $x + y \in V$, $\lambda x \in V$. (*V is closed under addition and scalar multiplication.*)
(ii) $x + y = y + x$, $x + (y + z) = (x + y) + z$. (*Addition is commutative and associative.*)
(iii) There exists a *zero vector* $0 \in V$ such that $x + 0 = 0 + x = x$ for all $x \in V$.
(iv) Each $x \in V$ has an *additive inverse* $-x \in V$ such that $x + (-x) = (-x) + x = 0$.
(v) $\lambda(x + y) = \lambda x + \lambda y$. (*Addition is distributive w.r.t. scalar multiplication.*)
(vi) $\lambda_1(\lambda_2 x) = (\lambda_1 \lambda_2)x$. (*Scalar multiplication is associative.*)

$\Diamond$

We have already encountered some vector spaces over $\mathbb{C}$. Here are some examples.

EXAMPLE B.2. The set $\mathbb{C}^n = \{(z_1, z_2, \ldots, z_n) : z_j \in \mathbb{C}\}$ of n-tuples of complex numbers, with the usual component-by-component addition and scalar multiplication. $\Diamond$

EXAMPLE B.3. The collections of functions on $\mathbb{T}$ we have been working with (Section 2.1) with the usual pointwise addition and scalar multiplication: $C(\mathbb{T})$, $C^k(\mathbb{T})$, $C^\infty(\mathbb{T})$; the collection $\mathcal{R}(\mathbb{T})$ of Riemann-integrable functions on $\mathbb{T}$; the Lebesgue spaces $L^p(\mathbb{T})$, in particular $L^1(\mathbb{T})$, $L^2(\mathbb{T})$, and $L^\infty(\mathbb{T})$; and the collection $\mathcal{P}_N(\mathbb{T})$ of 2π-periodic trigonometric polynomials of degree less than or equal to N. See the ladder of functional spaces in $\mathbb{T}$ in Figure 2.4. $\Diamond$

EXAMPLE B.4. The corresponding collections of functions on $\mathbb{R}$ discussed in the book: $C(\mathbb{R})$, $C^k(\mathbb{R})$, $C^\infty(\mathbb{R})$; the collection $C_c(\mathbb{R})$ of compactly supported continuous functions; similarly $C_c^k(\mathbb{R})$ and $C_c^\infty(\mathbb{R})$; the Lebesgue spaces $L^p(\mathbb{R})$;

the collection $\mathcal{S}(\mathbb{R})$ of Schwartz functions; infinitely often differentiable and rapidly decaying functions all of whose derivatives also are rapidly decaying; see Section 7.2 for precise definitions. ◊

EXAMPLE B.5. The space $\ell^2(\mathbb{Z})$ of sequences defined by equation (5.1) in Section 5.2, with componentwise addition and scalar multiplication. See also Example B.12. ◊

B.2. Normed spaces

It is useful to be able to measure closeness in a vector space. One way to do so is via a norm.

DEFINITION B.6. A *norm* over a vector space V over $\mathbb{C}$ (called a *normed vector space*) is a function $\|\cdot\| : V \to \mathbb{R}$ such that for all x, $y \in V$, $\alpha \in \mathbb{C}$:

(i) $\|x\| \geq 0$,
(ii) $\|x\| = 0$ if and only if $x = 0$,
(iii) $\|\alpha x\| = |\alpha|\,\|x\|$,
(iv) It satisfies the *triangle inequality*

$$\|x + y\| \leq \|x\| + \|y\|. \tag{B.1}$$

◊

Here are some examples of normed spaces.

EXAMPLE B.7. The set $\mathbb{R}$ with the Euclidean norm. The set $\mathbb{C}$ with norm given by the absolute value function. ◊

EXAMPLE B.8. The set $\mathbb{R}^n$ with norm the ℓ^p-norm, defined for each $x = (x_1, x_2, \ldots, x_n)$, and for each $1 \leq p < \infty$, by

$$\|x\|_p = \left(\sum_{k=1}^{n} |x_k|^p \right)^{1/p}.$$

◊

In finite-dimensional spaces *all norms are equivalent.* In particular, for $\mathbb{R}^n$ and the ℓ^p-norms there are positive constants A, $B > 0$ depending only on p and q such that

$$A\|x\|_p \leq \|x\|_q \leq B\|x\|_p.$$

The situation is different for infinite-dimensional spaces.

EXAMPLE B.9. The continuous functions over an interval I, or over $\mathbb{R}$, with the uniform norm. ◊

EXAMPLE B.10. The continuous functions over a closed interval I, with the L^p-norm. ◊

B.3. Inner-product vector spaces

Some vector spaces V come equipped with an extra piece of structure, called an inner product.

DEFINITION B.11. A (*strictly positive-definite*) *inner product* on a vector space V over $\mathbb{C}$ is a function $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{C}$ such that, for all x, y, $z \in V$, α, $\beta \in \mathbb{C}$,

(i) $\langle x, y\rangle = \overline{\langle y, x\rangle}$. (*The inner product is* skew-symmetric *or* Hermitian.)
(ii) $\langle \alpha x + \beta y, z\rangle = \alpha\langle x, z\rangle + \beta\langle y, z\rangle$. (*Linear in the first variable.*)
(iii) $\langle x, x\rangle \geq 0$. (*Positive-definite.*)
(iv) $\langle x, x\rangle = 0 \Longleftrightarrow x = 0$. ((iii) *&* (iv) *together: strictly positive-definite.*)

A vector space with such an inner product is called an *inner-product vector space.* $\Diamond$

Notice that (i) and (ii) imply that the inner product is *conjugate-linear* in the second variable, in other words

$$\langle z, \alpha x + \beta y\rangle = \overline{\alpha}\langle z, x\rangle + \overline{\beta}\langle y, x\rangle.$$

Every inner-product vector space V can be made into a normed space, since the quantity

$$\|x\| := \langle x, x\rangle^{1/2}$$

is always a norm on V, as we ask you to verify in Exercise B.20. This norm is known as the *induced norm* on V.

Here are some examples of inner-product vector spaces over $\mathbb{C}$.

EXAMPLE B.12. $V = \mathbb{C}^d$. The elements of $\mathbb{C}^d$ look like $z = (z_1, z_2, \ldots, z_n)$, with $z_j \in \mathbb{C}$. The inner product and its induced norm are defined by

$$\begin{aligned}
\langle z, w\rangle &:= z_1\overline{w_1} + z_2\overline{w_2} + \cdots + z_n\overline{w_n}, \\
\|z\|_{\mathbb{C}^n} &:= \langle z, z\rangle^{1/2} = (|z_1|^2 + |z_2|^2 + \cdots + |z_n|^2)^{1/2}.
\end{aligned}$$

$\Diamond$

EXAMPLE B.13. (*Little ℓ^2 of the Integers, over* $\mathbb{C}$) $V = \ell^2(\mathbb{Z})$. The elements of $\ell^2(\mathbb{Z})$ are doubly infinite sequences $\{a_n\}_{n\in\mathbb{Z}} = (\ldots, a_{-n}, \ldots, a_{-1}, a_0, a_1, \ldots, a_n, \ldots)$, with $a_n \in \mathbb{C}$, and $\sum_{n\in\mathbb{Z}} |a_n|^2 < \infty$. The inner product and its induced norm are defined for $x = \{a_n\}_{n\in\mathbb{Z}}$ and $y = \{b_n\}_{n\in\mathbb{Z}}$ in ℓ^2 by

$$\begin{aligned}
\langle x, y\rangle &:= \sum_{n=-\infty}^{\infty} a_n\overline{b_n}, \\
\|x\|_{\ell^2(\mathbb{Z})} &:= \langle x, x\rangle^{1/2} = \left(\sum_{n=-\infty}^{\infty} |a_n|^2\right)^{1/2}.
\end{aligned}$$

$\Diamond$

EXAMPLE B.14. $V = L^2(\mathbb{T})$. The inner product and its induced norm are defined by

$$\begin{aligned}
\langle f, g\rangle &:= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\overline{g(x)}\,dx, \\
\|f\|_{L^2(\mathbb{T})} &:= \langle f, f\rangle^{1/2} = \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} |f(x)|^2\,dx\right)^{1/2}.
\end{aligned}$$

$\Diamond$

The fact that the infinite sums in Example B.13 and the integrals in Example B.14 yield finite quantities is not automatic. It is a direct consequence of the indispensable *Cauchy–Schwarz Inequality*, which holds for all inner-product vector spaces and which we state, together with two other important inequalities, in Section B.3.1.

Had we considered $\mathcal{R}(\mathbb{T})$, the set of Riemann-integrable functions on $\mathbb{T}$, with the L^2-inner product and norm, we would not have encountered the difficulty with the integrals, since functions in $\mathcal{R}(\mathbb{T})$ are bounded, and so the integrals are automatically finite. However we would still have had the problem with the infinite sums, since boundedness, or even decay to zero, do not guarantee the convergence of the series.

B.3.1. Three key results involving orthogonality. *Orthogonality* is one of the most important ideas in the context of inner-product vector spaces.

In the two-dimensional case of the vector space $V = \mathbb{R}^2$ over $\mathbb{R}$, the usual inner product has a very natural geometric interpretation. Given two vectors $x = (x_1, x_2)$, $y = (y_1, y_2)$ in $\mathbb{R}^2$,

$$\langle x, y\rangle := x_1\overline{y_1} + x_2\overline{y_2} = \|x\|\,\|y\|\cos\theta,$$

where θ is the angle between the two vectors. In this setting, two vectors are said to be perpendicular or *orthogonal* when $\theta = \pi/2$, which happens if and only if $\langle x, y\rangle = 0$.

Let V be a vector space over $\mathbb{C}$, with inner product $\langle\cdot,\cdot\rangle$, and induced norm $\|\cdot\|$.

Definition B.15. Two vectors $x, y \in V$ are *orthogonal* if $\langle x, y\rangle = 0$. We use the notation $x \perp y$. A set of vectors $\{x_\lambda\}_{\lambda\in\Lambda} \subset V$ is *orthogonal* if $x_{\lambda_1} \perp x_{\lambda_2}$ for all $\lambda_1 \neq \lambda_2$, $\lambda_i \in \Lambda$. Two subsets X, Y of V are *orthogonal*, denoted by $X \perp Y$, if $x \perp y$ for all $x \in X$ and $y \in Y$. ♢

In Chapter 5 we show that the trigonometric functions $\{e_n(\theta) := e^{in\theta}\}_{n\in\mathbb{Z}}$ are *orthonormal* in $L^2(\mathbb{T})$. In other words, not only are they an orthogonal set, but each element in the set has norm one (the vectors have been *normalized*). Checking orthonormality is equivalent to verifying that

$$\langle e_n, e_m\rangle = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{in\theta}e^{-im\theta}\,d\theta = \begin{cases} 1, & \text{if } k = m; \\ 0, & \text{if } k \neq m. \end{cases}$$

Exercise B.16. Define vectors $e_k \in \mathbb{C}^N$ by $e_k(n) = (1/\sqrt{N})\, e^{2\pi ikn/N}$, for k, $n \in \{0, 1, 2, \ldots, N-1\}$. Show that the vectors $\{e_k\}_{k=0}^{N-1}$ form an orthonormal family in $\mathbb{C}^N$. ♢

The Pythagorean Theorem, the Cauchy–Schwarz Inequality, and the triangle inequality are well known in the two-dimensional case described above. They also hold in every inner-product vector space V over $\mathbb{C}$.

Theorem B.17 (Pythagorean Theorem). *If x, $y \in V$ are orthogonal, then*

$$\|x + y\|^2 = \|x\|^2 + \|y\|^2.$$

Theorem B.18 (Cauchy–Schwarz Inequality). *For all x, y in V,*

$$|\langle x, y\rangle| \leq \|x\|\|y\|.$$

Theorem B.19 (Triangle Inequality). *For all x, $y \in V$,*

$$\|x + y\| \leq \|x\| + \|y\|.$$

Exercise B.20. Prove Theorems B.17, B.18, and B.19, first for the model case $V = \mathbb{R}^2$, and then for a general inner-product vector space V over $\mathbb{C}$. In particular, show that $\|x\| := \sqrt{\langle x, x\rangle}$ is a norm in V. ♢

Exercise B.21. Deduce from the Cauchy–Schwarz Inequality that if f and g are in $L^2(\mathbb{T})$, then $|\langle f, g\rangle| < \infty$. Similarly, if $x = \{a_n\}_{n\in\mathbb{Z}}$, $y = \{b_n\}_{n\in\mathbb{Z}} \in \ell^2(\mathbb{Z})$, then $|\langle x, y\rangle| < \infty$. ♢

Exercise B.22. Suppose that V is an inner-product vector space, Λ is an arbitrary index set, and $\{x_\lambda\}_{\lambda\in\Lambda}$ is an orthogonal family of vectors. Show that these vectors are *linearly independent*; in other words, for any finite subset of indices $\{\lambda_1, \dots, \lambda_n\} \subset \Lambda$ such that n is no larger than the dimension of V (which may be infinite),

$$a_1 x_{\lambda_1} + \dots + a_n x_{\lambda_n} = 0 \quad \text{if and only if} \quad a_1 = \dots = a_n = 0.$$

♢

B.3.2. Orthonormal bases. Orthogonality implies *linear independence* (see Exercise B.22), and if our geometric intuition is right, orthogonality is in some sense the most linearly independent a set could be. In Exercise B.16 we found a collection of N vectors in the N-dimensional vector space $\mathbb{C}^N$. Since these N vectors are linearly independent, they constitute a *basis* of $\mathbb{C}^N$. Moreover the set $\{e_k\}_{k=0}^{N-1}$ is an *orthonormal basis* of $\mathbb{C}^N$. We will call it the *N-dimensional Fourier basis.* In Chapter 6 we point out the analogies between this finite-dimensional theory, which only requires linear algebra facts, and the discrete Fourier theory.

The trigonometric functions $\{e_n(\theta) := e^{in\theta}\}_{n\in\mathbb{Z}}$ are also an orthonormal basis of $L^2(\mathbb{T})$, but it takes some time to verify this fact. See Chapter 5. Having an infinite orthonormal family (hence a linearly independent family) tells us that the space is infinite-dimensional. So far, there is no guarantee that there are no other functions in the space orthogonal to the given orthonormal family, or in other words that *the orthonormal system is complete.*

Definition B.23 (Complete system/orthonormal basis). Let $\{x_n\}_{n\in\mathbb{N}}$ be an orthonormal family in an inner-product vector space V over $\mathbb{C}$. We say that the family is *complete*, or that $\{x_n\}_{n\in\mathbb{N}}$ is a *complete system*, or that the vectors in the family form an *orthonormal basis*, if given any vector $x \in V$, x can be expanded into a series of the basis elements which is convergent in the norm induced by the inner product. That is, there exists a sequence of complex numbers $\{a_n\}_{n\in\mathbb{N}}$ such that

$$\lim_{N\to\infty} \Big\| x - \sum_{n=1}^{N} a_n\, x_n \Big\| = 0.$$

Equivalently, $x = \sum_{n=1}^{\infty} a_n x_n$ where equality holds in the norm of V.

The coefficients a_n of x are uniquely determined. They can be calculated by pairing the vector with the basis elements:

$$a_n = \langle x, x_n\rangle.$$

♢

How can we tell whether a given orthonormal family X in a vector space V is actually an orthonormal basis for V? One criterion for completeness is that the only vector orthogonal to all the vectors in X is the zero vector. Another is that Plancherel's Identity must hold. These ideas are summarized in Theorem B.24, which holds for all inner-product vector spaces V over $\mathbb{C}$. In Chapter 5 we prove Theorem B.24 for the particular case of $V = L^2(\mathbb{T})$ with the trigonometric basis. Showing that the same proofs extend to the general case is left as an exercise.

THEOREM B.24. *Let V be an inner-product vector space over $\mathbb{C}$ and $X = \{x_n\}_{n\in\mathbb{N}}$ an orthonormal family in V. Then the following are equivalent:*

(i) *X is a complete system, hence an orthonormal basis of V.*
(ii) *If $x \perp X$ then $x = 0$.*
(iii) (Plancherel's Identity, or Infinite-Dimensional Pythagorean Theorem) *For all $x \in V$,*

$$\|x\|^2 = \sum_{n=1}^{\infty} |\langle x, x_n\rangle|^2.$$

B.3.3. Banach and Hilbert spaces. Some, but not all, inner-product vector spaces satisfy another property, to do with convergence of Cauchy sequences of vectors. Namely, they are *complete inner-product vector spaces*, also known as *Hilbert spaces*. For example, $L^2(\mathbb{T})$ is a Hilbert space, but $\mathcal{R}(\mathbb{T})$ with the L^2-inner product is not a Hilbert space.

First note that since we have the norm induced by the inner product, we can certainly talk about sequences of vectors converging to another vector. Specifically if V is a vector space with a norm (a *normed space*), then we say a sequence $\{x_n\} \in V$ *converges to* $x \in V$ if

$$\lim_{n\to\infty} \|x_n - x\| = 0.$$

It is well known that if a sequence is converging to a point, then as $n \to \infty$ the vectors in the sequence are getting closer to each other, in the sense that they form a *Cauchy sequence*.

DEFINITION B.25. The sequence $\{x_n\} \in V$ is *Cauchy* if for every $\varepsilon > 0$ there exists $N > 0$ such that $\|x_n - x_m\| < \varepsilon$ for all n, $m > N$. ♢

The converse is not always true, except in so-called *complete normed spaces* or *Banach spaces*. For instance, the Cauchy sequence 1, 1.4, 1.41, 1.414, 1.4142, ... in the space $\mathbb{Q}$ of rational numbers does not converge to any point in $\mathbb{Q}$; it does converge to a point in the larger vector space $\mathbb{R}$. Furthermore, *every* Cauchy sequence of real numbers converges to some real number. $\mathbb{Q}$ is not complete; $\mathbb{R}$ is complete.

DEFINITION B.26. A *Banach space* is a complete normed space $\mathcal{B}$. In other words, every sequence in $\mathcal{B}$ that is Cauchy with respect to the norm of $\mathcal{B}$ converges to a limit in $\mathcal{B}$. ♢

EXAMPLE B.27. The continuous functions on a closed and bounded interval, with the uniform norm, form a Banach space. This fact and its proof are part of an advanced calculus course; for example see [**Tao2**, Theorem 14.4.5]. ♢

EXAMPLE B.28. The spaces $L^p(I)$ defined in Chapter 2 are Banach spaces. The proof of this fact belongs to a more advanced course. However it is crucial for us, and we assume this result without proof. ♢

Some of these spaces have the additional geometric structure of an inner product, as is the case for $L^2(\mathbb{T})$.

DEFINITION B.29. A *Hilbert space* $\mathcal{H}$ is an inner-product vector space such that $\mathcal{H}$ is complete. In other words, every sequence in $\mathcal{H}$ that is Cauchy with respect to the norm induced by the inner product of $\mathcal{H}$ converges to a limit in $\mathcal{H}$. ♢

Notice that here we are using the word "complete" with a different meaning than that in Definition B.23, where we consider "complete systems of orthonormal functions". The context indicates whether we are talking about a "complete space" or a "complete system $\{f_n\}$ of functions".

Here are some canonical examples of Hilbert spaces.

Example B.30. C^n, a finite-dimensional Hilbert space. $\Diamond$

Example B.31. $\ell^2(\mathbb{Z})$, an infinite-dimensional Hilbert space. $\Diamond$

Example B.32. $L^2(\mathbb{T})$, the Lebesgue square-integrable functions on $\mathbb{T}$, consisting of the collection $\mathcal{R}(\mathbb{T})$ of all Riemann square-integrable functions on $\mathbb{T}$, together with all the functions that arise as limits of Cauchy sequences in $\mathcal{R}(\mathbb{T})$ (similarly to the construction of the real numbers from the rational numbers). In other words, $L^2(\mathbb{T})$ is the *completion* of $\mathcal{R}(\mathbb{T})$ with respect to the L^2-metric. $\Diamond$

Example B.33. $L^2(\mathbb{R})$, the Lebesgue square-integrable functions on $\mathbb{R}$. $\Diamond$