Harmonic Analysis: from Fourier to Haar

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CHAPTER 3

Pointwise convergence of Fourier series

In this chapter we study pointwise and uniform convergence of Fourier series for smooth functions. We also discuss the connection between the decay of the Fourier coefficients and smoothness. We end with a historical account of convergence theorems.

We assume that the reader is familiar with the notions of pointwise and uniform convergence. In Chapter 2 we review these concepts, as well as a number of other modes of convergence that arise naturally in harmonic analysis.

3.1. Pointwise convergence: why do we care?

For the periodic ramp function in Example 1.12, the Fourier expansion converges *pointwise* to $f(\theta) = \theta$ everywhere except at the odd multiples of π, where it converges to zero (the halfway height in the vertical jump). For the toy model of a voice signal in Example 1.1, we were able to reconstruct the signal perfectly using only four numbers (two frequencies and their corresponding amplitudes), because the signal was very simple. In practice, complicated signals such as real voice signals transmitted through phone lines require many more amplitude coefficients and frequencies for accurate reconstruction, sometimes thousands or millions. Should we use all of these coefficients in reconstructing our function? Or perhaps we could get a reasonable approximation to our original signal using only a subset of the coefficients? If so, which ones should we keep? One simple approach is to truncate at specific frequencies $-N$ and N, and to use only the coefficients corresponding to frequencies such that $|n| \leq N$. Instead of the full Fourier series, we obtain the N^{th} *partial Fourier sum of f*,

$$S_N f(\theta) = \sum_{|n| \leq N} a_n e^{in\theta} = \sum_{|n| \leq N} \widehat{f}(n) e^{in\theta}, \tag{3.1}$$

which is a trigonometric polynomial of degree N.

Is $S_N f$ like f? In general, we need to know whether $S_N f$ looks like f as $N \to \infty$, then judge which N will be appropriate for truncation so that the reconstruction is accurate enough for the purposes of the application at hand.

For the periodic ramp function f in Example 1.12, the partial Fourier sums are

$$S_N f(x) = \sum_{|n| \leq N} \frac{(-1)^{n+1}}{in} e^{in\theta} = 2 \sum_{n=1}^{N} \frac{(-1)^{n+1}}{n} \sin(n\theta).$$

In Figure 3.1 we plot f together with $S_N f$, for $N = 2$, 5, and 20.

Exercise 3.1. Use the following Matlab script [***script has to be made] to reproduce Figure 3.1. Experiment with other values of N. Describe what you see. Near the jump discontinuity you will see an overshoot that remains large as

FIGURE 3.1. Plots of the periodic ramp function f and its partial Fourier sums $S_N f$, for $N = 2$, 5, and 20.

N increases. This is the so-called *Gibbs*[1] *phenomenon*. Repeat the exercise for the *periodic step function*

$$g(\theta) = \begin{cases} 1, & \text{if } \theta \in [0, \pi); \\ -1, & \text{if } \theta \in [-\pi, 0). \end{cases}$$

Project 12.4 outlines a more detailed investigation of the Gibbs phenomenon. ◊

EXERCISE 3.2. Compute the partial Fourier sums for the trigonometric polynomial in Exercise 1.11. ◊

Here is an encouraging sign that pointwise approximation by partial Fourier sums may work well for many functions. As the example in Exercise 3.2 suggests, if f is a 2π-periodic trigonometric polynomial of degree M, in other words a function of the form

$$f(\theta) = \sum_{|k| \le M} a_k e^{ik\theta},$$

then for $N \ge M$, its partial Fourier sums $S_N f(x)$ coincide with $f(x)$. Therefore $S_N f$ converges pointwise to f, for all θ and for all trigonometric polynomials f. *A fortiori*, the convergence is uniform on $\mathbb{T}$, since $S_M f = S_{M+1} f = \cdots = f$. See Exercise 1.18.

The partial Fourier sums $S_N f(\theta)$ sums converge uniformly to $f(\theta)$ for all trigonometric polynomials f.

EXERCISE 3.3. Verify that the Taylor polynomials $P_N(f, 0)$ converge uniformly for all polynomials f. See (1.4) and Exercise 1.5. ◊

Approximating with partial Fourier sums presents some problems. For example, there are continuous functions for which one does not get pointwise convergence everywhere, as du Bois-Reymond[2] showed. One of the deepest theorems in twentieth century analysis, due to Lennart Carleson[3], says that one does get *almost everywhere* convergence for *square-integrable functions on* $\mathbb{T}$, and in particular for continuous functions; see Section 3.3 for more details, and Chapter 2 for definitions.

One can obtain better results in approximating a function f if one is allowed to combine the Fourier coefficients in ways different from the partial sums $S_N f$. For example, averaging over the partial Fourier sums provides a smoother truncation method in terms of the *Cesáro*[4] *means*,

$$\sigma_N f(\theta) = \frac{S_0 f(\theta) + S_1 f(\theta) + \cdots + S_{N-1} f(\theta)}{N}, \tag{3.2}$$

[1] Named after the American mathematician Josiah Willard Gibbs (1839–1903), who reported it in a letter to *Nature* in 1899. The Gibbs phenomenon has to do with how poorly a Fourier series converges near a jump discontinuity of the function f. More precisely, for each N there is a neighborhood of the discontinuity, decreasing in size as N increases, on which the partial sums $S_N f$ *always overshoot the left- and right-hand limits of f by about 9%.*

[2] The German mathematician Paul David Gustav du Bois-Reymond (1831–1889).

[3] Carleson was born in Stöckholm, Sweden in 1928. He was awarded the Abel Prize in 2006 "for his profound and seminal contributions to harmonic analysis and the theory of smooth dynamical systems".

[4] Named after the Italian mathematician Ernesto Cesáro (1859–1906).

that we will study more carefully in Section 4.5. In particular, we will prove Fejér's[5] Theorem 4.28 that we quote now.

> *The Cesáro means $\sigma_N f$ converge uniformly to f for continuous functions f on $\mathbb{T}$.*

Therefore the following *uniqueness principle* holds:

> *If $f \in C(\mathbb{T})$ and $\widehat{f}(n) = 0$ for all n, then $f = 0$.*

Just observe that if $\widehat{f}(n) = 0$ for all n then $\sigma_N f = 0$ for all $N > 0$, but the Cesáro means $\sigma_N f$ converge uniformly to f. Therefore f must be identically equal to zero.

The Cesáro means are themselves trigonometric polynomials, so we can deduce from this result a celebrated theorem named after the German mathematician Karl Theodor Wilhelm Weierstrass (1815–1897).

Theorem 3.4 (Weierstrass's Approximation Theorem for Trigonometric Polynomials). *Continuous functions on $\mathbb{T}$ can be approximated uniformly by trigonometric polynomials. Equivalently, the trigonometric functions are dense in the continuous functions on $\mathbb{T}$ with respect to the uniform norm.*

Aside 3.5. *There is a version of Weierstrass's Theorem for continuous functions on closed and bounded intervals and plain polynomials. Namely, continuous functions on $[a, b]$ can be approximated uniformly by polynomials. Equivalently, the polynomials are dense in $C([a, b])$ with respect to the uniform norm. See* [**Tao2**, Section 14.8, Theorem 14.8.3]. *Both versions are special cases of a more general result, the Stone*[6]*–Weierstrass Theorem, which can be found in more advanced textbooks such as* [**Fol**, Chapter 4, Section 7].

To sum up, continuous periodic functions on the circle can be uniformly approximated by trigonometric polynomials, but if we insist on approximating a continuous periodic function f by the particular trigonometric polynomials given by its partial Fourier sums $S_N f$, then even pointwise convergence can fail. However, it turns out that if we assume that f is smoother, we will get both pointwise and uniform convergence.

3.2. Smoothness vs convergence

In this section we present a first convergence result for functions that have at least two continuous derivatives. In the proof we obtain some decay of the Fourier coefficients which can be generalized to the case when the functions are k times differentiable. We then explore whether decay of the Fourier coefficients implies smoothness, and we study the rate of convergence of the partial Fourier sums for smooth functions.

3.2.1. A first convergence result. Let us consider a function $f \in C^2(\mathbb{T})$, so f is 2π-periodic and twice continuously differentiable. We show that the numbers $S_N f(\theta)$ do converge, and that they converge to $f(\theta)$, for each $\theta \in \mathbb{T}$. In the following theorem we show that the partial sums converge; afterwards we argue that the limit must be $f(\theta)$.

[5]Named after the Hungarian mathematician Lipót Fejér (1880–1959).

[6]Named after the American mathematician Marshall Harvey Stone (1903–1989).

THEOREM 3.6. *Let $f \in C^2(\mathbb{T})$. Then for each $\theta \in \mathbb{T}$ the limit of the partial sums*

$$\lim_{N\to\infty} S_N f(\theta) =: \sum_{n=-\infty}^{\infty} \widehat{f}(n) e^{in\theta}$$

exists. Moreover, the convergence to the limit is uniform on $\mathbb{T}$.

PROOF. The partial Fourier sums of f are given by

$$S_N f(\theta) = \sum_{|n|\le N} \widehat{f}(n) e^{in\theta}.$$

Fix θ, and notice that the Fourier series of f evaluated at θ is a power series in the variable $z = e^{i\theta}$:

$$\sum_{n=-\infty}^{\infty} \widehat{f}(n) e^{in\theta} = \sum_{n=-\infty}^{\infty} \widehat{f}(n) \left(e^{i\theta}\right)^n.$$

It suffices to show that this numerical series is absolutely convergent, or in other words that the series of its absolute values is convergent, that is

$$\sum_{n=-\infty}^{\infty} |\widehat{f}(n) e^{in\theta}| = \sum_{n=-\infty}^{\infty} |\widehat{f}(n)| < \infty,$$

because then the Fourier series itself must converge, and the convergence is uniform, by the Weierstrass M-Test, see Theorem 2.51.

We can find new expressions for the Fourier coefficients $\widehat{f}(n)$, for $n \neq 0$, by integrating by parts twice:

$$\begin{aligned}
\widehat{f}(n) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)e^{-in\theta}\,d\theta \qquad (\text{by parts with } u = f(\theta),\ v = -\frac{1}{in}e^{-in\theta})\\
&= \frac{1}{2\pi}\left[-\frac{1}{in} f(\theta) e^{-in\theta}\right]_{-\pi}^{\pi} - \frac{1}{2\pi}\int_{-\pi}^{\pi} -\frac{1}{in} f'(\theta)e^{-in\theta}\,d\theta\\
&= \frac{1}{2\pi}\frac{1}{in}\int_{-\pi}^{\pi} f'(\theta)e^{-in\theta}\,d\theta\\
&= \frac{1}{in}\widehat{f'}(n)\\
&= -\frac{1}{n^2}\widehat{f''}(n) \qquad (\text{by the same argument applied to } f').
\end{aligned}$$

The integration by parts is permissible since f is assumed to have two continuous derivatives. The boundary term in the second line vanishes because of the continuity and 2π-periodicity of both f and the trigonometric functions (hence of their product). We conclude that for $n \neq 0$,

$$\widehat{f'}(n) = in\widehat{f}(n), \qquad \widehat{f''}(n) = -n^2\widehat{f}(n). \tag{3.3}$$

We are now ready to make a direct estimate of the coefficients, for $n \neq 0$,

$$\begin{aligned}
|\widehat{f}(n)| &= \frac{1}{n^2}|\widehat{f''}(n)|^2\\
&= \frac{1}{n^2}\left|\frac{1}{2\pi}\int_{-\pi}^{\pi} f''(\theta)e^{-in\theta}\,d\theta\right|\\
&\le \frac{1}{n^2}\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} |f''(\theta)|\,d\theta\right),
\end{aligned}$$

since $|e^{-in\theta}| = 1$. But f'' is continuous on the closed interval $[-\pi, \pi]$. So it is bounded by a finite constant $C > 0$, and therefore its average over any bounded interval is also bounded by the same C: $\frac{1}{2\pi}\int_{-\pi}^{\pi} |f''(\theta)|\, d\theta \le C$. We conclude that there exists a constant $C > 0$ such that

$$|\widehat{f}\,(n)| \le \frac{C}{n^2}, \qquad \text{for all } n \neq 0.$$

Therefore, by the comparison test, $\sum_{n=-\infty}^{\infty} |\widehat{f}\,(n)|$ converges, and by the *Weierstrass M-Test* (Theorem 2.51), the Fourier series $\sum_{n=-\infty}^{\infty} \widehat{f}\,(n)e^{inx}$ does converge uniformly and hence pointwise if $f \in C^2(\mathbb{T})$. □

ASIDE 3.7. *The equations* (3.3) *are instances of a* time–frequency dictionary *that we summarize in Table 4.1, in page 69. Specifically,* derivatives of f are transformed into multiplication of $\widehat{f}$ by polynomials. *As a consequence, linear differential equations are transformed into algebraic equations.*

We can restate Theorem 3.6 as follows.

If $f \in C^2(\mathbb{T})$, then its Fourier series converges uniformly on $\mathbb{T}$.

REMARK 3.8. We have not yet established whether the Fourier series evaluated at θ converges to $f(\theta)$ or to some other number.

If $f \in C^2(\mathbb{T})$, it is true that f is the pointwise limit of its Fourier series, though it takes more work to prove this result directly. It is a consequence of the following elementary lemma about complex numbers.

LEMMA 3.9. *Suppose that the sequence of complex numbers $\{a_n\}_{n\ge 1}$ converges. Then the sequence of its averages also converges, and their limits coincide; that is*

$$\lim_{n\to\infty} \frac{a_1 + a_2 + \cdots + a_n}{n} = \lim_{n\to\infty} a_n.$$

EXERCISE 3.10. Prove Lemma 3.9. Give an example to show that if the sequence of averages converges, that does not imply that the original sequence converges. ♢

In Section 4.5 we show that for continuous functions f on $\mathbb{T}$, the Cesáro means $\sigma_N f(\theta)$, see (3.2), which are averages of the partial Fourier sums, do converge uniformly to $f(\theta)$ as $N \to \infty$, see Theorem 4.28. Lemma 3.9 shows that since for $f \in C^2(\mathbb{T})$ the partial Fourier sums converge (Theorem 3.6), then the Cesáro sums also converge, and to the same limit. It follows that if $f \in C^2(\mathbb{T})$, then the partial sums $S_N f(\theta)$ of the Fourier series for f do converge pointwise to $f(\theta)$, for each $\theta \in \mathbb{T}$.

Summing up, we have established the following theorem (modulo Fejér's Theorem 4.28).

THEOREM 3.11. *If $f \in C^2(\mathbb{T})$, then its Fourier series converges uniformly to f on $\mathbb{T}$.*

In fact, the partial Fourier sums converge uniformly for C^1-functions, but the argument we just gave does not suffice to guarantee pointwise convergence, since we can no longer argue by comparison, as we just did, to obtain absolute convergence. In this case, the estimate via integration by parts on the absolute value of the coefficients is insufficient; all we get is $|a_n| \leq C/n$, and the corresponding series does not converge. A more delicate argument is required just to check convergence of the sequence $\{S_N f(\theta)\}$ for $f \in C^1(\mathbb{T})$. But once the convergence of the series is guaranteed, then the same result about convergence of Cesáro sums for continuous functions applies, and the limit has no other choice than to coincide with $f(\theta)$. The following result is proved in [**Str**, Chapter 12, Theorem 12.2.2].

THEOREM 3.12. *If $f \in C^1(\mathbb{T})$, then its Fourier series converges uniformly on $\mathbb{T}$ to f.*

3.2.2. Smoothness vs rate of decay of Fourier coefficients. In Section 3.2.1, we integrated twice by parts and obtained formulae (3.3) for the Fourier coefficients of the first and second derivatives of a C^2-function. If the function is C^k, then we can iterate the procedure k times, obtaining a formula for the Fourier coefficients of the k^{th} derivative of f in terms of the Fourier coefficients of f. Namely, for $n \neq 0$,

$$\widehat{f^{(k)}}(n) = (in)^k \widehat{f}(n).$$

Furthermore, we obtain an estimate about the rate of decay of the Fourier coefficients of f if the function is k times continuously differentiable on $\mathbb{T}$, for $n \neq 0$,

$$|\widehat{f}(n)| \leq \frac{C}{n^k}.$$

Here we have used once more the fact that $f^{(k)}$ is bounded on the closed interval $[-\pi, \pi]$, hence its Fourier coefficients are bounded: $|\widehat{f^{(k)}}(n)| < C$. What we just have shown is the following,

THEOREM 3.13. *If $f \in C^k(\mathbb{T})$, then its Fourier coefficients $\widehat{f}(n)$ decay at least like n^{-k}.*

If the Fourier coefficients of an integrable function decay like n^{-k} is it true that f is C^k? No. It is not even true that f is C^{k-1}. Consider the following example when $k = 1$.

EXAMPLE 3.14. The *characteristic function* of the interval $[0, 1]$ on $\mathbb{T}$ is defined to be 1 for all $\theta \in [0, 1]$ and zero on $\mathbb{T} \setminus [0, 1]$, it is denoted $\chi_{[0,1]}$. Its Fourier coefficients are given by

$$\widehat{\chi}_{[0,1]}(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \chi_{[0,1]}(\theta) e^{-in\theta}\, d\theta = \frac{1}{2\pi} \int_0^1 e^{-in\theta}\, d\theta = \frac{1 - e^{-in}}{2\pi in}.$$

Clearly $|\widehat{\chi}_{[0,1]}(n)| \leq (\pi n)^{-1}$, so the Fourier coefficients decay like n^{-1}. However, the characteristic function is not continuous. $\Diamond$

One can construct similar examples where the Fourier coefficients decay like n^{-k}, $k \geq 2$, and the function is not in $C^{k-1}(\mathbb{T})$. However, it follows from Corollary 3.16 that such function must be in $C^{k-2}(\mathbb{T})$ for $k \geq 2$.

THEOREM 3.15. *Let $f : \mathbb{T} \to \mathbb{C}$ be 2π-periodic and integrable, $\ell \geq 0$. If $\sum_{n \in \mathbb{Z}} |\widehat{f}(n)|\, |n|^{\ell} < \infty$, then f is C^{ℓ}.*

Furthermore the partial Fourier sums converge uniformly to f, and the derivatives up to order ℓ of the partial Fourier sums converge uniformly to the corresponding derivatives of f.

COROLLARY 3.16. *If* $|\widehat{f}(n)| \leq C|n|^{-k}$ *for* $k \geq 2$, *and* $n \neq 0$, *then* f *is* C^ℓ, *where* $\ell = k-2$ *if* $k \in \mathbb{Z}$, *and* $\ell = [k]-1$ *otherwise. Here* $[k]$ *denotes the* integer part of k, *in other words the largest integer less than or equal to* k.

EXERCISE 3.17. Deduce Corollary 3.16 from Theorem 3.15. ◊

We sketch the proof of Theorem 3.15.

PROOF OF THEOREM 3.15. For $\ell = 0$, the hypothesis is $\sum_{n\in\mathbb{Z}} |\widehat{f}(n)| < \infty$, which in turn implies the uniform convergence of the partial Fourier sums $S_N f$ to some periodic and continuous function F. Féjer's Theorem 4.28 shows that the averages of the partial Fourier sums, the Cesàro means $\sigma_N f$, see (3.2), converge uniformly to $f \in C(\mathbb{T})$, finally Exercise 3.10 implies that $F = f$, hence the partial Fourier sums converge uniformly to f.

For the case $\ell = 1$, see Exercise 3.18.

Now prove the general case by induction on ℓ. □

EXERCISE 3.18. Let $f : \mathbb{T} \to \mathbb{C}$ be continuous. Assume that $\sum_{n\in\mathbb{Z}} |\widehat{f}(n)|\,|n|$ converges. Show that $S_N f \to f$ uniformly, and that $(S_N f)' \to h$ uniformly for some function h. Show that f is C^1 and that

$$\lim_{N\to\infty} \sum_{|n|\leq N} in\widehat{f}(n)e^{in\theta} = f'(\theta).$$

In other words, $h = f'$. Hint: Theorem 2.52 about interchanging limits and differentiation might be useful. ◊

In this section we learned that the Fourier coefficients of smooth functions go to zero as $|n| \to \infty$. The question is, at what rate? We found that

> *The smoother the function, the faster the rate of decay of its Fourier coefficients.*

What if f is merely continuous? Is it still true that the Fourier coefficients decay to zero? The answer is yes.

LEMMA 3.19 (Riemann–Lebesgue Lemma for continuous functions). *Let* $f \in C(\mathbb{T})$. *Then*

$$\lim_{|n|\to\infty} \widehat{f}(n) = 0.$$

PROOF. Observe that because $e^{\pi i} = -1$, we can write the n^{th} Fourier coefficient of $f \in C(\mathbb{T})$ in a slightly different form, namely

$$\begin{aligned}
\widehat{f}(n) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)e^{-in\theta}\,d\theta \\
&= -\frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)e^{-in\theta}e^{\pi i}\,d\theta \\
&= -\frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)e^{-in(\theta-\pi/n)}\,d\theta \\
&= -\frac{1}{2\pi}\int_{-\pi-\pi/n}^{\pi-\pi/n} f\left(\alpha+\frac{\pi}{n}\right)e^{-in\alpha}\,d\alpha,
\end{aligned}$$

where we have made the change of variable $\alpha = \theta - \frac{\pi}{n}$. But since the integrand is 2π-periodic, and we are integrating over an interval of length 2π, we can shift the integral back to $\mathbb{T}$ without altering its value, and conclude that

$$\widehat{f}(n) = -\frac{1}{2\pi}\int_{-\pi}^{\pi} f\left(\theta + \frac{\pi}{n}\right) e^{-in\theta}\, d\theta.$$

Averaging the original integral representation of $\widehat{f}(n)$ with this new representation gives yet another integral representation, which involves a difference of values of f, and for which we can use the hypothesis that f is continuous. More precisely,

$$\widehat{f}(n) = \frac{1}{4\pi}\int_{-\pi}^{\pi} \left[f(\theta) - f\left(\theta + \frac{\pi}{n}\right)\right] e^{-in\theta}\, d\theta.$$

Hence

$$|\widehat{f}(n)| \leq \frac{1}{4\pi}\int_{-\pi}^{\pi} \left|f(\theta) - f\left(\theta + \frac{\pi}{n}\right)\right| d\theta.$$

We can now take the limit as $|n| \to \infty$, and since f is uniformly continuous on $[-\pi, \pi]$, the sequence $g_n(\theta) := \left|f(\theta) - f\left(\theta + \frac{\pi}{n}\right)\right|$ converges uniformly to zero. Therefore we can interchange the limit and the integral (see Theorem 2.49), obtaining

$$0 \leq \lim_{|n|\to\infty} |\widehat{f}(n)| \leq \frac{1}{4\pi}\int_{-\pi}^{\pi} \lim_{|n|\to\infty} \left|f(\theta) - f\left(\theta + \frac{\pi}{n}\right)\right| d\theta = 0,$$

as required. □

Exercise 3.20. Show that if f is continuous and 2π-periodic, then it is uniformly continuous on $[-\pi, \pi]$. Moreover, the functions $g_n(\theta) := \left|f(\theta) - f\left(\theta + \frac{\pi}{n}\right)\right|$ converge uniformly to zero. ♢

We will encounter other versions of the Riemann–Lebesgue Lemma in subsequent chapters.

3.2.3. Differentiability vs rate of convergence of partial Fourier sums. It can be shown that

> *The smoother f is, the faster is the rate of convergence of the partial Fourier sums to f.*

More precisely, it can be proven that if $f \in C^k(\mathbb{T})$ for $k \geq 2$, then there exists a constant $C > 0$ independent of θ such that

$$|S_N f(\theta) - f(\theta)| \leq \frac{C}{N^{k-1}}. \tag{3.4}$$

If $f \in C^1(\mathbb{T})$, then there exists a constant $C > 0$ independent of θ such that

$$|S_N f(\theta) - f(\theta)| \leq \frac{C}{\sqrt{N}}. \tag{3.5}$$

These results automatically provide uniform convergence of the partial Fourier sums for C^k-functions, $k \geq 1$. See [**Str**, Thm 12.2.2] for the proofs of these results.

If a function satisfies inequality (3.4) for $k \geq 2$, it does not necessarily mean that f is C^{k-1}. Therefore f is not necessarily C^k, as is shown in Example 3.21 for $k = 2$. It is not difficult to modify that example for the case $k > 2$.

EXAMPLE 3.21. (*Plucked String*) Consider the function $f : \mathbb{T} \to \mathbb{R}$ given by $f(\theta) = \pi/2 - |\theta|$, whose graph is shown in Figure 3.2. This function is continuous but not differentiable at $\theta = 0, \pm\pi$. Its Fourier coefficients are given by

$$\widehat{f}(n) = \begin{cases} 0, & \text{if } n \text{ is even;} \\ 2/(\pi n^2), & \text{if } n \text{ is odd.} \end{cases}$$

In particular $\sum_{n\in\mathbb{Z}} |\widehat{f}(n)| < \infty$, and so by Theorem 3.15, $S_N f \to f$ uniformly on $\mathbb{T}$. Furthermore, the following estimates hold:

$$|S_N f(\theta) - f(\theta)| \leq \frac{2\pi^{-1}}{N-1}, \qquad |S_N f(0) - f(0)| \geq \frac{2\pi^{-1}}{N+2}. \tag{3.6}$$

The second estimate implies that the rate of convergence cannot be improved uniformly in θ. ♢

EXERCISE 3.22. Compute the Fourier coefficients of the plucked string function in Example 3.21. Verify that the inequalities (3.6) hold. Hint: It might help to remember the following inequalities

$$\frac{1}{n} - \frac{1}{n+1} = \frac{1}{n(n+1)} \leq \frac{1}{n^2} \leq \frac{1}{n(n-1)} = \frac{1}{n-1} - \frac{1}{n}.$$

The left- and right-hand sides in the string of inequalities, when added up, form telescoping series. This observation allows you to obtain the following bounds:

$$\frac{1}{N} \leq \sum_{n\geq N} \frac{1}{n^2} \leq \frac{1}{N-1}.$$

If this hint is not enough, see [**Kor**, Example 10.1, p.35]. ♢

FIGURE 3.2. The plucked string function f.

3.3. A suite of convergence theorems

Here we state, without proofs and in chronological order, the main results concerning pointwise convergence of the Fourier series for continuous functions and for Lebesgue-integrable functions. The Fourier series converges to $f(\theta)$ for functions f that are slightly more than just continuous at θ, but not necessarily differentiable there. In this suite of theorems we try to illustrate how delicate the process of mathematical thinking can be, with a number of intermediate results and puzzling examples, which slowly help bring to light the right concepts and the most complete and satisfactory results.

This might be a good time to revisit Chapter 2. In this section we will use notation and concepts introduced in Chapter 2, such as sets of measure zero, almost everywhere convergence, Lebesgue-integrable functions and convergence in L^p.

Project 12.6 investigates in more detail some of the results below.

THEOREM 3.23 (Dirichlet, 1829). *Suppose that, except perhaps at finitely many points, $f : \mathbb{T} \to \mathbb{C}$ is continuous and its derivative f' is continuous and bounded. Then for all $\theta \in \mathbb{T}$,*

$$\lim_{N\to\infty} S_N f(\theta) = \frac{f(\theta^+) + f(\theta^-)}{2}, \tag{3.7}$$

where $f(\theta^+)$ is the limit when α approaches θ from the right, and $f(\theta^-)$ is the limit when α approaches θ from the left. In particular, $S_N f(\theta) \to f(\theta)$ as $N \to \infty$ at all points $\theta \in \mathbb{T}$ where f is continuous.

Thinking wishfully, we might dream that the partial Fourier sums for a continuous function would converge everywhere. That dream was shattered by the following result.

THEOREM 3.24 (Du Bois-Reymond, 1873). *There is a continuous function $f : \mathbb{T} \to \mathbb{C}$ such that*

$$\limsup_{N\to\infty} |S_N f(0)| = \infty.$$

Here the partial sums of the Fourier series of a continuous function fail to converge at the point $x = 0$. The modern proof of this result uses the Uniform Boundedness Principle which we will encounter later in the book in Section 9.3.4. For a constructive proof see [**Kor**, Chapter 18, Theorem 18.1]; see also [**SS1**, Chapter 3, Section 2.2].

A reproduction of Dirichlet's paper can be found in [**KL**, Chapter 4], as well as a comparative analysis with Jordan's[7] result that we quote next.

THEOREM 3.25 (Jordan, 1881). *If f is a function of* bounded variation[8] *on $\mathbb{T}$, then the Fourier series of f converges to $\frac{1}{2}[f(\theta^+) + f(\theta^-)]$ at every point $\theta \in \mathbb{T}$, and in particular to $f(\theta)$ at every point of continuity. The convergence is uniform on closed intervals of continuity.*

The class of functions that have bounded variation in I is denoted by $BV(I)$. Bounded variation implies Riemann integrability, that is $BV(I) \subset \mathcal{R}(I)$. The conditions in Theorem 3.23 imply bounded variation, and so Dirichlet's Theorem follows from Jordan's Theorem. A proof of Dirichlet's Theorem can be found in [**Kor**, Chapter 16, p.59]. A similar proof works for Jordan's Theorem. The following result, Dini's Criterion, can be used to deduce Dirichlet's Theorem, at least at points of continuity.

THEOREM 3.26 (Dini, 1878). *Let f be a 2π-periodic function. Suppose that for some θ there exists a $\delta > 0$ such that*

$$\int_{|t|<\delta} \left| \frac{f(\theta+t) - f(\theta)}{t} \right| dt < \infty;$$

in particular f must be continuous at θ. Then

$$\lim_{N\to\infty} S_N f(\theta) = f(\theta).$$

A 2π-periodic function f is said to satisfy a *uniform Lipschitz or Hölder condition of order α*, for $0 < \alpha \le 1$, if there exists $C > 0$ such that $|f(\theta+t)-f(\theta)| \le C|t|^\alpha$ for all t and θ. The class of 2π-periodic functions that satisfy a uniform Lipschitz

[7]Camille Jordan, French mathematician (1838–1922).

[8]This means that the *total variation* on $\mathbb{T}$ of f is bounded. The total variation of f is defined to be

$$V(f) = \sup_P \sum_{n=1}^{N} |f(\theta_n) - f(\theta_{n-1})|,$$

where the supremum is taken over all partitions $P : \theta_0 = -\pi < \theta_1 < \cdots < \theta_{n-1} < \theta_n = \pi$ of $\mathbb{T}$. Step functions are functions of bounded variation, so are monotone functions, and you can compute precisely their variation.

condition of order α is denoted by $C^\alpha(\mathbb{T})$. It is clear that $C(\mathbb{T}) \subset C^\alpha(\mathbb{T}) \subset C^\beta(\mathbb{T})$ for all $0 < \alpha \leq \beta \leq 1$. If $f \in C^\alpha(\mathbb{T})$ for $0 < \alpha \leq 1$, then it satisfies uniformly Dini's condition at all $\theta \in \mathbb{T}$, and hence the partial Fourier sums converge pointwise to f. If $\alpha > 1/2$ and $f \in C^\alpha(\mathbb{T})$ the convergence of the partial Fourier sums is absolute [**SS1**, Chapter 3, Exercises 15–16]. A proof of Dini's result can be found in [**Graf**, Theorem 3.3.6, p.189].

These criteria depend only on the values of f in an arbitrarily small neighborhood of θ. This is a general and surprising fact known as the *Riemann Localization Principle*, which was part of a Memoir presented by Riemann in 1854, but only published after his death in 1867. See [**KL**, Chapter 5, Sections 1 and 4].

THEOREM 3.27 (Riemann Localization Principle). *For an integrable function f, the convergence of the Fourier series to $f(\theta)$ only depends on the values of f in a neighborhood of θ.*

At the beginning of the twentieth century, pointwise convergence everywhere of the Fourier series was ruled out for periodic and continuous functions. However positive results could be obtained by slightly improving the continuity. In this chapter we proved such a result for smooth (C^2) functions, and we have just stated a much refined result where the improvement over continuity is encoded in Dini's condition. Can one have a theorem guaranteeing pointwise convergence everywhere for a larger class than the class of Dini-continuous functions? Where can we draw the line?

In 1913, the Russian mathematician Nikolai Nikolaevich Luzin (1883–1950) conjectured that: *Every square-integrable function, and thus in particular every continuous function, equals the sum of its Fourier series almost everywhere.*

Ten years later Andrey Nikolaevich Kolmogorov (1903–1987), another famous Russian mathematician, found a Lebesgue-integrable function whose partial Fourier sums *diverge almost everywhere.* Three years later he constructed an even more startling example [**Kol**].

THEOREM 3.28 (Kolmogorov, 1926). *There is a Lebesgue-integrable function $f : \mathbb{T} \to \mathbb{C}$ such that*

$$\limsup_{N\to\infty} |S_N f(\theta)| = \infty \qquad \textit{for all } \theta \in \mathbb{T}.$$

Recall that the minimal requirement on f so that we can compute its Fourier coefficients is that f is Lebesgue-integrable $\big(f \in L^1(\mathbb{T})\big)$. In that case $S_N f$ is a well-defined trigonometric polynomial for each $N \in \mathbb{N}$. Kolmogorov's result tells us that we can cook up a Lebesgue-integrable function whose partial Fourier sums $S_N f$ diverge at *every* point θ! His function is Lebesgue integrable, but it is not continuous; in fact it is essentially unbounded on every interval.

After Kolmogorov's example, experts believed that it was only a matter of time before the same sort of example could be constructed for a continuous function. It came as a big surprise when, half a century later, Carleson proved Luzin's conjecture, which implies that the partial Fourier sums of every continuous function converge at almost every points θ. Carleson's result is one of the deepest theorems in analysis in the twentieth century.

THEOREM 3.29 (Carleson, 1966). *Suppose $f : \mathbb{T} \to \mathbb{C}$ is square-integrable. Then $S_N f(\theta) \to f(\theta)$ as $N \to \infty$, except possibly on a set of measure zero.*

In particular, Carleson's conclusion holds if f is continuous on $\mathbb{T}$, or more generally if f is Riemann-integrable on $\mathbb{T}$.

This result implies that Kolmogorov's example cannot be a square-integrable function, in particular it cannot be Riemann integrable. Du Bois-Reymond's example does not contradict Carleson's Theorem. Although the function is square integrable (since it is continuous), the partial Fourier series are allowed to diverge on a set of measure zero (which can be large, as we mentioned in Section 2.1.4).

The next question is whether, given a set of measure zero in $\mathbb{T}$, one can construct a continuous function whose Fourier series diverges exactly on the given set. The next theorem answers this question.

Theorem 3.30 (Kahane[9] and Katznelson[10], 1966). *If $E \subset \mathbb{T}$ is a set of measure zero, then there is a continuous function $f : \mathbb{T} \to \mathbb{C}$ such that*

$$\limsup_{N\to\infty} |S_N f(\theta)| = \infty \qquad \text{for all } \theta \in E.$$

The proof of Theorem 3.30 can be found in Katznelson's book *An introduction to harmonic analysis* [**Kat**, Chapter II, Sec 3]. The same chapter includes a construction of a Kolmogorov-type example.

A year later the following non-trivial extension of Carleson's Theorem was proved by the American mathematician Richard Hunt.

Theorem 3.31 (Hunt, 1967). *Carleson's Theorem holds for all $f \in L^p(\mathbb{T})$ and for all $p > 1$.*

It took almost 40 years for the Carleson–Hunt Theorem to make its way into graduate textbooks in full detail. There are now several more accessible accounts of this famous theorem, for instance in [**Graf**] and in [**Ari**].

Now we see where to draw the line between those functions for which the partial Fourier series converge pointwise for all x to $f(x)$, and the ones for which they do not. For example, $f \in C^k(\mathbb{T})$ is enough, but f being continuous is not. However, for continuous functions we will get convergence almost everywhere, and given any set of measure zero there is a continuous function whose partial Fourier series diverges on that set.

As Körner writes [**Kor**, p.75],

> *The problem of pointwise convergence is thus settled. There are few questions which have managed to occupy even a small part of humanity for 150 years. And of those questions, very few indeed have been answered with as complete and satisfactory an answer as Carleson has given to this one.*

[9] Jean-Pierre Kahane, French mathematician, 1926–.

[10] Yitzhak Katznelson, Israeli mathematician, 1934–.