1. Introduction

\[ S_N f(x) := \sum_{|n| \leq N} \hat{f}(n) e^{inx} := \sum_{|n| \leq N} \left( \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx} \, dx \right) e^{inx} \]

denotes the partial Fourier sum for \( f \) at the point \( x \)

Can use Fourier Sums to approximate \( f : \mathbb{T} \to \mathbb{R} \) under the appropriate conditions
1. Introduction

- **Def**: Pointwise Convergence of Functions:

  A series of functions \( \{f_n\}_{n \in \mathbb{N}} \) converges pointwise to \( f \) on the set \( D \) if \( \forall \epsilon > 0 \) and \( \forall x \in D \), \( \exists N \in \mathbb{N} \) such that \( |f_n(x) - f(x)| < \epsilon \forall n \geq N \).

- **Def**: Uniform Convergence of Functions:

  A series of functions \( \{f_n\}_{n \in \mathbb{N}} \) converges uniformly to \( f \) on the set \( D \) if \( \forall \epsilon > 0 \), \( \exists N \in \mathbb{N} \) such that \( |f_n(x) - f(x)| < \epsilon \forall n \geq N \) and \( \forall x \in D \).

**Figure**: Pointwise Conv.

**Figure**: Uniform Conv.
Some movies to illustrate the difference. (First Uniform then Pointwise)
1. Introduction

- Some movies to illustrate the difference. (First Uniform then Pointwise)

- Persistent “bump” in second illustrates Gibbs’ Phenomenon.
2. Gibbs’ Phenomenon: A Brief History

Basic Background

- 1848: Property of overshooting discovered by Wilbraham
- 1899: Gibbs brings attention to behavior of Fourier Series (Gibbs observed same behavior as Wilbraham but by studying a different function)
- 1906: Maxime Brocher shows that the phenomenon occurs for general Fourier Series around a jump discontinuity
Key Players and Contributions

- Lord Kelvin: Constructed two machines while studying tide heights as a function of time, \( h(t) \).
  - Machine capable of computing periodic function \( h(t) \) using Fourier Coefficients
  - Constructed a “Harmonic Analyzer” capable of computing Fourier coefficients of past tide height functions, \( h(t) \).

- A. A. Michelson:
  - Elaborated on Kelvin’s device and constructed a machine “which would save considerable time and labour involved in calculations. . . of the resultant of a large number of simple harmonic motions.”
  - Using first 80 coefficients of sawtooth wave, Michelson’s machine closely approximated the sawtooth function except for two blips near the points of discontinuity
2. Gibbs’ Phenomenon: A Brief History

Key Players and Contributions Cont.

- Wilbraham

  - 1848: Wilbraham investigated the equation:

    \[ y = \cos(x) + \frac{\cos(3x)}{3} + \frac{\cos(5x)}{5} + \ldots + \frac{\cos((2n-1)x)}{2n-1} + \ldots \]

  - Discovered that “at a distance of \(\frac{\pi}{4}\) alternately above and below it, joined by perpendiculars which are themselves part of the locus” the equation overshoots \(\frac{\pi}{4}\) by a distance of:

    \[ \frac{1}{2} \left| \int_{\pi}^{\infty} \frac{\sin(x)}{x} \, dx \right| \]

  - Also suggested that similar analysis of

    \[ y = 2 \left( \sin(x) - \frac{\sin(2x)}{2} + \frac{\sin(3x)}{3} - \ldots + (-1)^{n+1} \frac{\sin(n+1)x)}{n} \right) \]  

    (An equation explored later on) would lead to an analogous result

  - These discoveries gained little attention at the time
J. Willard Gibbs

1898: Published and article in *Nature* investigating the behavior of the function given by:

\[ y = 2 \left( \sin(x) - \frac{\sin(2x)}{2} + \frac{\sin(3x)}{3} - \ldots + (-1)^{n+1} \frac{\sin(nx)}{n} \right) \]

Gibbs observed in this first article that the limiting behavior of the function (*sawtooth*) had “vertical portions, which are bisected the axis of \( X \), [extending] beyond the points where they meet the inclined portions, their total lengths being express by four times the definite integral \( \int_{0}^{\pi} \frac{\sin(u)}{u} \, du \).”
Key Players and Contributions Cont.

- Brocher
  - 1906: In an article in *Annals of Mathematics*, Brocher demonstrated that Gibbs’ Phenomenon will be observed in any Fourier Series of a function $f$ with a jump discontinuity saying that the limiting curve of the approximating curves has a vertical line that “has to be produced beyond these points by an amount that bears a definite ratio to the magnitude of the jump.”
  - Before Brocher, Gibbs’ Phenomenon had only been observed in specific series without any generalization of the concept.
Consider the *square wave* function given by:

\[ g(x) := \begin{cases} 
-1 & \text{if } -\pi \leq x < 0 \\
1 & \text{if } 0 \leq x < \pi 
\end{cases} \]

**Goal:** Analytically prove that Gibbs’ Phenomenon occurs at jump discontinuities of this $2\pi$-periodic function
Using the definition $a_n := \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)e^{-inx} \, dx$, basic Calculus yields

$$S_N f(x) = \sum_{n=-N}^{N} a_n e^{in\theta} = \frac{1}{\pi} \sum_{|n| \leq N} \left( \frac{2}{n} \right) \frac{e^{in\theta}}{i}$$

$$= \frac{4}{\pi} \sum_{m=1}^{M} \frac{\sin((2m-1)\theta)}{2m-1}$$

where $M$ is the largest integer such that $2M - 1 \leq N$. 
Here we notice that

\[
\frac{\sin((2m - 1)\theta)}{2m - 1} = \int_0^x \cos((2m - 1)\theta) \, d\theta
\]

which yields the following identity:

\[
S_{2M-1}g(x) = \frac{4}{\pi} \sum_{m=1}^{M} \frac{\sin((2m - 1)x)}{2m - 1}
\]

\[
= \frac{4}{\pi} \sum_{m=1}^{M} \int_0^x \cos((2m - 1)\theta) \, d\theta
\]

\[
= \frac{4}{\pi} \int_0^x \sum_{m=1}^{M} \cos((2m - 1)\theta) \, d\theta
\]
Now using the identity

$$\sum_{m=1}^{M} \cos((2m - 1)x) = \frac{\sin(2Mx)}{2 \sin(x)}$$

we get the following:

$$S_{2M-1}g(x) = \frac{4}{\pi} \int_0^x \sum_{m=1}^{M} \cos((2m - 1)\theta) \, d\theta$$

$$= \frac{4}{\pi} \int_0^x \frac{\sin(2M\theta)}{2 \sin(\theta)} \, d\theta$$
At this point, using basic analysis of the derivative of $S_{2M-1}g(x)$, we see that there are local maximum and minimum values at

$$x_{M,+} = \frac{\pi}{2M} \quad \text{and} \quad x_{M,-} = -\frac{\pi}{2M}$$

Here, using the substitution $u = 2M\theta$ and the fact that for $u \approx 0$, $\sin(u) \approx u$, we get that for large $M$

$$S_{2M-1}g(x_{m,+}) \approx \frac{2}{\pi} \int_{0}^{\pi} \sin(\theta) \frac{\theta}{\theta} d\theta$$

$$\text{and}$$

$$S_{2M-1}g(x_{m,-}) \approx -\frac{2}{\pi} \int_{0}^{\pi} \sin(\theta) \frac{\theta}{\theta} d\theta$$

Further, the value $\frac{2}{\pi} \int_{0}^{\pi} \sin(\theta) \frac{\theta}{\theta} d\theta \approx 1.17898$. Have shown that

$\forall M >> 1$, $M \in \mathbb{N}$, $\exists x = x_{M,+}$ such that $S_{2M-1}g(x_{m,+}) \approx 1.17898$. \Rightarrow is always a “blip” near the jump discontinuity, namely at $x = \frac{\pi}{2M}$, where the value $S_{2M-1}g(x)$ is not near 1!
Have shown analytically that a “blip” persists for all $S_{2M-1}g$ near the jump discontinuity.

Demonstrated again by square wave
Have shown analytically that a “blip” persists for all $S_{2M-1}g$ near the jump discontinuity.

Demonstrated again by square wave

We can also see the same behavior in the $2\pi$-periodic sawtooth function given by $f(x) = x$ on the interval $[-\pi, \pi)$. 
For both of the *square wave* and *sawtooth* functions, we have shown that a “blip” of constant size persists

⇒ while $S_M g \rightarrow g$ pointwise, we now know that $S_M g \nrightarrow g$ uniformly since for small enough neighborhoods, the “blip” is always outside the boundaries. (revisit square wave animation)
Kernels And Convolutions

**Kernels:**

**Def**: (Good Kernel) A family \( \{K_n\}_{n \in \mathbb{N}} \) real-valued integrable functions on the circle is a *family of good kernels* if the following hold:

1. **(i)** \( \forall n \in \mathbb{N}, \frac{1}{2\pi} \int_{-\pi}^{\pi} K_n(\theta) \, d\theta = 1. \)
2. **(ii)** \( \exists M > 0 \) such that \( \forall n \in \mathbb{N}, \int_{-\pi}^{\pi} |K_n(\theta)| \, d\theta \leq M. \)
3. **(iii)** \( \forall \delta > 0, \int_{\delta \leq |\theta| < \pi} |K_n(\theta)| \, d\theta \to 0 \) as \( n \to \infty. \)
Kernels And Convolutions

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  3. \( \forall \delta > 0, \int_{\delta \leq |\theta| < \pi} |K_n(\theta)| \, d\theta \to 0 \) as \( n \to \infty. \)

- **Dirichlet:** \( D_N(\theta) = \sum_{|n| \leq N} e^{in\theta} \)
Kernels And Convolutions

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  - **Def**: (Good Kernel) A family \( \{K_n\}_{n \in \mathbb{N}} \) real-valued integrable functions on the circle is a *family of good kernels* if the following hold:

  \[
  \begin{align*}
  \text{(i)} & \quad \forall n \in \mathbb{N}, \quad \frac{1}{2\pi} \int_{-\pi}^{\pi} K_n(\theta) \, d\theta = 1. \\
  \text{(ii)} & \quad \exists M > 0 \text{ such that } \forall n \in \mathbb{N}, \quad \int_{-\pi}^{\pi} |K_n(\theta)| \, d\theta \leq M. \\
  \text{(iii)} & \quad \forall \delta > 0, \quad \int_{\delta \leq |\theta| < \pi} |K_n(\theta)| \, d\theta \to 0 \text{ as } n \to \infty.
  \end{align*}
  \]

- **Dirichlet:** \( D_N(\theta) = \sum_{|n| \leq N} e^{in\theta} \)

\[
\Rightarrow \int_{-\pi}^{\pi} D_N(\theta) \, d\theta = \sum_{|n| \leq N} \int_{-\pi}^{\pi} e^{in\theta} \, d\theta
\]

Recall: Dirichlet Kernel is \textit{NOT} a good kernel and Fejér Kernel is.
Kernels And Convolutions

- **Kernels:**
  - **Def**: (Good Kernel) A family \( \{K_n\}_{n \in \mathbb{N}} \) real-valued integrable functions on the circle is a *family of good kernels* if the following hold:

  1. \( \forall n \in \mathbb{N}, \quad \frac{1}{2\pi} \int_{-\pi}^{\pi} K_n(\theta) \, d\theta = 1. \) *(ii)* \( \exists M > 0 \) such that \( \forall n \in \mathbb{N}, \quad \int_{-\pi}^{\pi} |K_n(\theta)| \, d\theta \leq M. \) *(iii)* \( \forall \delta > 0, \quad \int_{\delta \leq |\theta| < \pi} |K_n(\theta)| \, d\theta \to 0 \) as \( n \to \infty. \)

- **Dirichlet:** \( D_N(\theta) = \sum_{|n| \leq N} e^{in\theta} \)

  \[
  \Rightarrow \int_{-\pi}^{\pi} D_N(\theta) \, d\theta = \sum_{|n| \leq N} \int_{-\pi}^{\pi} e^{in\theta} \, d\theta
  \]

- **Fejér:** \( F_N(\theta) = [D_0(\theta) + D_1(\theta) + \ldots + D_{N-1}(\theta)]/N \)
Kernels And Convolutions

- **Kernels:**
  - **Def**: (Good Kernel) A family \( \{K_n\}_{n \in \mathbb{N}} \) real-valued integrable functions on the circle is a *family of good kernels* if the following hold:

  1. \( \forall n \in \mathbb{N}, \frac{1}{2\pi} \int_{-\pi}^{\pi} K_n(\theta) \, d\theta = 1 \).
  2. \( \exists M > 0 \) such that \( \forall n \in \mathbb{N}, \int_{-\pi}^{\pi} |K_n(\theta)| \, d\theta \leq M \).
  3. \( \forall \delta > 0, \int_{\delta \leq |\theta| < \pi} |K_n(\theta)| \, d\theta \to 0 \) as \( n \to \infty \).

- **Dirichlet:** \( D_N(\theta) = \sum_{|n| \leq N} e^{in\theta} \)

  \[ \Rightarrow \int_{-\pi}^{\pi} D_N(\theta) \, d\theta = \sum_{|n| \leq N} \int_{-\pi}^{\pi} e^{in\theta} \, d\theta \]

  - **Fejér:** \( F_N(\theta) = [D_0(\theta) + D_1(\theta) + \ldots + D_{N-1}(\theta)]/N \)
  - **Recall:** Dirichlet Kernel is *NOT* a good kernel and Fejér Kernel is...
Kernels And Convolutions Cont.

- Convolution:
  - **Def**: The convolution of two functions, $f$ and $g$ is given by:

$$ (f \ast g)(\theta) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(y)g(\theta - y) \, dy $$

(1)
Kernels And Convolutions Cont.

Convolution:
- **Def**: The convolution of two functions, $f$ and $g$ is given by:

$$ (f * g)(\theta) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(y)g(\theta - y) \, dy $$

(1)

$$ \Rightarrow S_M f = (D_M * f)(\theta) $$
Kernels And Convolutions Cont.

- Convolutions:
  - **Def**: The convolution of two functions, $f$ and $g$ is given by:
    \[
    (f \ast g)(\theta) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(y)g(\theta - y) \, dy
    \]  
    (1)
  
    \[\Rightarrow S_M f = (D_M \ast f)(\theta)\]

- **Def**: \(\sigma_M f(\theta) := [S_0 f(\theta) + S_1 f(\theta) + ... + S_{M-1} f(\theta)]/M\)
**Kernels And Convolutions Cont.**

- **Convolution:**
  - **Def**: The convolution of two functions, \( f \) and \( g \) is given by:

\[
(f * g)(\theta) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(y)g(\theta - y) \, dy \tag{1}
\]

\[
\Rightarrow S_M f = (D_M * f)(\theta)
\]

- **Def**: \( \sigma_M f(\theta) := \frac{1}{M} [S_0 f(\theta) + S_1 f(\theta) + ... + S_{M-1} f(\theta)] \)

- From a previous homework, we also know that \( F_M * f = \sigma_M f \)
Kernels And Convolutions Cont.

**Convolution:**

**Def**: The convolution of two functions, $f$ and $g$ is given by:

\[
(f \ast g)(\theta) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(y)g(\theta - y) \, dy
\]

\[ (1) \]

\[ \Rightarrow S_M f = (D_M \ast f)(\theta) \]

**Def**: $\sigma_M f(\theta) := \left[ S_0 f(\theta) + S_1 f(\theta) + \ldots + S_{M-1} f(\theta) \right] / M$

From a previous homework, we also know that $F_M \ast f = \sigma_M f$

\[ \Rightarrow F_M \ast f \text{ is of sort an averaging of the Dirichlet kernels, or more importantly, the partial Fourier sums.} \]
The Problem is Quite General

- Brocher showed Gibbs’ Phenomenon occurs with any $2\pi$-periodic piecewise-continuous function with a jump discontinuity.
- Overshoot is proportional to height of jump:
  - Square Wave: Overshoot $\approx 0.17898$
  - $\Rightarrow 0.17898 = k(2)$
  - $\Rightarrow k = 0.08949$
  - $\Rightarrow$ Overshoot/Undershoot given by $\approx 0.08949(f(x_d^+) - f(x_d^-))$ where $x_d$ is point of discontinuity
  - Suggests an overshoot of $\approx 0.08949(2\pi) \approx 0.56228$ for a maximum height of $\approx 3.70228$

- Thus, need a way to resolve this for modeling of periodic phenomena to be practical.
- Using Fejér Kernel averages Dirichlet kernels and could therefore be a way of eliminating this phenomenon.
- animation experiment
Gibbs’ Phenomenon observed in all $2\pi$-periodic functions with a jump discontinuity.

In some cases, it is quite basic to show its persistence analytically.

Causes nastiness in approximations of functions by Fourier series, but is easily remedied with *averaging* using Fejér Kernel.