A Bowl of Kernels

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When we studied Fourier series we improved convergence by convolving with the Fejer and Poisson kernels on $\mathbb{T}$. The analogous Fejer and Poisson kernels on the real line help us improve convergence of the Fourier integral. In this project we define the Fejer kernel, the Poisson kernel, the heat kernel, the Dirichlet kernel, and the conjugate Poisson kernel. We study some of their properties, applications, and connections with complex variables.
Fejer Kernel on $\mathbb{R}$

\[ F_R(x) = R \left( \frac{\sin \pi R x}{\pi R x} \right)^2 \quad R > 0 \quad (1) \]

The Poisson kernel on $\mathbb{R}$

\[ P_y(x) = \frac{y}{\pi (x^2 + y^2)} \quad y > 0 \quad (2) \]

The Heat Kernel on $\mathbb{R}$

\[ H_t(x) = e^{\frac{-|x|^2}{4t}} \quad t > 0 \quad (3) \]
The Dirichlet kernel on $\mathbb{R}$

$$D_R(x) = \int_{-R}^{R} e^{2\pi i x \xi} d\xi = \frac{\sin(2\pi R x)}{\pi x} \quad R > 0 \quad (4)$$

The Conjugate Poisson kernel on $\mathbb{R}$

$$Q_y(x) = \frac{x}{\pi(x^2 + y^2)} \quad y > 0 \quad (5)$$
Poisson Kernel and Conjugate Poisson Kernel solve the Laplace equation. $\Delta P = (\frac{\partial^2}{\partial x^2})P + (\frac{\partial^2}{\partial y^2})P = 0,$

$\Delta Q = (\frac{\partial^2}{\partial x^2})Q + (\frac{\partial^2}{\partial y^2})Q = 0$ for $(x, y)$ in the upper half-plane $(x \in R, y > 0)$.

Proof (Poisson Kernel):

\[
\frac{\partial P}{\partial x} = \frac{y}{\pi} (-1)(x^2 + y^2)^{-2} 2x
\]

\[
\frac{\partial P}{\partial x^2} = \frac{8x^2y}{\pi(x^2 + y^2)^3} - \frac{2y}{\pi(x^2 + y^2)^2}
\]

\[
\frac{\partial P}{\partial y} = \frac{x^2 - y^2}{\pi(x^2 + y^2)^2}
\]

\[
\frac{\partial P}{\partial y^2} = \frac{-6yx^2 + 2y^3}{\pi(x^2 + y^2)^3}
\]
\[ \Delta P = \frac{8x^2 y}{\pi(x^2 + y^2)^3} - \frac{2y}{\pi(x^2 + y^2)^2} + \frac{-6yx^2 + 2y^3}{\pi(x^2 + y^2)^3} \]
\[ = \frac{2x^2 y + 2y^3 - 2y(x^2 + y^2)}{\pi(x^2 + y^2)^3} \]
\[ = 0 \]

**Theorem**

*Heat Kernel is a solution of the heat equation on the line.*

\[ (\frac{\partial}{\partial t}) H = (\frac{\partial^2}{\partial x^2}) H \]
In class we learned that an Approximation of the Identity on $\mathbb{R}$ is a family $\{K_t\}_{t \in \Lambda}$ where the following holds:

1. The functions $\{K_t\}_{t \in \Lambda}$ have mean value 1: $\int_{-\infty}^{\infty} K_t(x) dx = 1$
2. The functions are uniformly integrable in $t$: There is a constant $C \geq 0$ such that $\int_{-\infty}^{\infty} |K_t(x)| dx \leq C$ for all $t \in \Lambda$
3. Concentration of mass at the origin: For each $\delta > 0$, $\lim_{t \to t_0} \int_{|x| \geq \delta} |K_t(x)| dx = 0$
Recall the Fejer Kernel is defined as,

$$F_R(x) = R \left( \frac{\sin \pi R x}{\pi R x} \right)^2 \quad R > 0$$  \hspace{1cm} (6)

Mean value 1:

$$\int_{-R}^{R} R \left( \frac{\sin \pi R x}{\pi R x} \right)^2 dx = 1$$  \hspace{1cm} (7)

Let $u = \pi R x$, \hspace{0.5cm} $dx = \frac{du}{\pi R}$ \hspace{1cm} Then we have,

$$\frac{1}{\pi} \int_{-R}^{R} \left( \frac{\sin u}{u} \right)^2 du = \frac{1}{\pi} \int_{-R}^{R} \frac{1 - \cos^2 u}{u^2} du$$  \hspace{1cm} (8)

$$= \frac{1}{\pi} \int_{-R}^{R} \frac{(1 - \cos u)(1 + \cos u)}{u^2} du$$  \hspace{1cm} (9)
Now we can use integration by parts with $u' = 1 + \cos u$ and $dv = \frac{1 - \cos u}{\pi u^2} du'$

Then we can use the well known identity, $\int_{R} \frac{1 - \cos x}{x^2} dx = \pi$ and we have,

$$= (1 + \cos u) + \int_{R} \sin u' du' = (1 + \cos u) - \cos u = 1 \quad (10)$$

Uniformly Integrable in $t$:
There is a constant $C \geq 0$ such that $\int_{-\infty}^{\infty} \left| R\left( \frac{\sin \pi R x}{\pi R x} \right)^2 \right| dx \leq C$ for all $R > 0$

But by 1, we know $\int_{R} R\left( \frac{\sin \pi R x}{\pi R x} \right)^2 dx = 1$ and also $R\left( \frac{\sin \pi R x}{\pi R x} \right)^2 \geq 0$

So just let $C = 1$. 

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Concentration of mass at the origin: For each \( \delta > 0 \)
\[
\lim_{R \to \infty} \int_{|x| \geq \delta} |R\left(\frac{\sin \pi R x}{\pi R x}\right)^2| \, dx = 0.
\]
Let \( \delta > 0 \),
We know
\[
R\left(\frac{\sin \pi R x}{\pi R x}\right)^2 \leq R\left(\frac{1}{(\pi x)^2}\right)^2 = \frac{1}{R(\pi x)^2}
\]
So we have,
\[
\int_{|x| \geq \delta} R\left(\frac{\sin \pi R x}{\pi R x}\right)^2 \leq 2 \int_{\delta}^{\infty} \frac{1}{R(\pi x)^2}
\]
\[
= 2 \lim_{L \to \infty} \frac{1}{R(\pi)^2 L} - \frac{1}{R(\pi)^2 \delta}
\]
\[
= \frac{2}{R(\pi)^2 \delta}
\]
Now we take the limit in \( R \),
\[
\lim_{R \to \infty} \frac{2}{R(\pi)^2 \delta} = 0
\]
**Definition**

A function $f$ is said to be a continuous function of moderate decrease if $f$ is continuous and if there are constants $A$ and $\epsilon > 0$ such that

$$|f(x)| \leq \frac{A}{(1+|x|^{1+\epsilon})} \quad \text{for all } x \in \mathbb{R}$$
Theorem

Neither the Poisson kernel nor the Fejer kernel belong to $S(R)$. For each $R > 0$ and each $y > 0$ the Fejer kernel $F_R$ and the Poisson kernel $P_y$ are continuous functions of moderate decrease with $\epsilon = 1$. The conjugate Poisson kernel $Q_y$ is not a continuous function of moderate decrease.

Proof (For Poisson Kernel) $P_y(x) = \frac{y}{\pi(x^2+y^2)}$ for $y > 0$

If $0 < y < 1$,

$$|P_y(x)| = \left| \frac{y}{\pi y^2((\frac{x}{y})^2 + 1)} \right|$$

$$= \frac{1}{(\pi y)(1 + \frac{x^2}{y^2})}$$
If $0 < y < 1$ then

\[
\begin{align*}
y^2 &< 1 \\
\frac{1}{y^2} &> 1 \\
x^2 &> x^2 \\
\frac{x^2}{y^2} + 1 &> x^2 + 1
\end{align*}
\]

Then

\[
|P_y(x)| = \frac{1}{(\pi y)(1 + \frac{x^2}{y^2})} < \frac{1}{\pi y(x^2 + 1)}
\]
If \( y \geq 1 \)

\[
\begin{align*}
  y & \geq 1 \\
  y^2 & \geq 1 \\
  x^2 + y^2 & \geq q x^2 + 1
\end{align*}
\]

Then

\[
|P_y(x)| = \left| \frac{y}{\pi y^2((\frac{x}{y})^2 + 1)} \right| \leq \frac{y}{\pi(x^2 + 1)}
\]

Hence,

\[
A_y = \begin{cases} 
  \frac{1}{\pi y} & \text{if } 0 < y < 1, \\
  \frac{y}{\pi} & \text{if } y \geq 1
\end{cases}
\]
Let $S_R f$ denote the partial Fourier integral of $f$, defined by

$$S_R f(x) = \int_{-R}^{R} \hat{f}(\xi) e^{2\pi i x \xi} d\xi$$  \hspace{1cm} (15)$$

Then $S_R f(x) = D_R * f$ and $F_R(x) = \frac{1}{R} \int_0^R D_t(x) dt$.

Proof:

$$D_R * f = \int_R \hat{D}_R(x - y) f(y) dy$$  \hspace{1cm} (16)$$

$$= \int_R \int_{-R}^{R} e^{2\pi (x - y) \xi} d\xi f(y) dy$$  \hspace{1cm} (17)$$

$$= \int_{-R}^{R} \int_R e^{-2\pi i y \xi} f(y) dy e^{2\pi i x \xi} d\xi = \int_{-R}^{R} \hat{f}(\xi) e^{2\pi i x \xi} d\xi$$  \hspace{1cm} (18)$$
Also, $F_R(x) = \frac{1}{R} \int_0^R D_t(x) \, dt$ because,

\[
\frac{1}{R} \int_0^R D_t(x) \, dt = \frac{1}{R} \int_0^R \frac{\sin(2\pi tx)}{\pi x} \, dt = \frac{1}{R} \left( \left. -\frac{1}{\pi x} \cos(2\pi tx) \right|_0^R \right) = \frac{1}{R} \frac{1}{R^2 \pi^2 x^2} (\cos(0) - \cos(2\pi Rx)) = \frac{(\sin(\pi Rx))^2}{R \pi^2 x^2} = F_R(x)
\]
We have the Fejer Kernel can be written as the integral mean of the Dirichlet kernel, and the Fejer kernel define an approximation to the identity on $\mathbb{R}$ as $R \to \infty$. Therefore, the integral Cesaro means of a function of moderate decrease converge to $f$ as $R \to \infty$.

Theorem

$$\sigma_R f(x) = \frac{1}{R} \int_0^R S_t f(x) dt = F_R * f(x) \to f(x)$$

(24)

The convergence is both uniform and in $L^p$
We will prove for $p=1$. We will show,

$$\lim_{R \to \infty} \|F_R \ast f - f\|_1 = 0$$  \hspace{1cm} (25)$$

$$|F_R(x) \ast f(x) - f(x)| = \left| \int_R f(x - y)F_R(y)dy - f(x) \right|$$  \hspace{1cm} (26)$$

$$= \left| \int_R (f(x - y) - f(x))F_R(y)dy \right|$$  \hspace{1cm} (27)$$

$$\leq \int_R |(f(x - y) - f(x))| |F_R(y)| dy$$  \hspace{1cm} (28)$$

$$\|F_R \ast f - f\|_1 \leq \int_R \int_R |(f(x - y) - f(x))| |F_R(y)| dydx$$  \hspace{1cm} (29)$$

(30)
By Fubini,

\[ \| F_R * f - f \|_1 \leq \int_R \|(f(x - y) - f(x))\|_1 |F_R(y)| \, dy \]  
\[ = \int_{|y| > \delta} \|(f(x - y) - f(x))\|_1 |F_R(y)| \, dy \]  
\[ + \int_{|y| < \delta} \|(f(x - y) - f(x))\|_1 |F_R(y)| \, dy \]  
\[ = l_1 + l_2 \]  

Notice, \( l_2 = \int_{|y| < \delta} \|(f(x - y) - f(x))\|_1 |F_R(y)| \, dy \) can be made arbitrarily small by continuity of \( f \).
Also, \( I_1 = \int_{|y| > \delta} \| (f(x - y) - f(x)) \|_1 |F_R(y)| \, dy \) can be made arbitrarily small since \( F_R \) is an approximation to the identity. This concludes the proof, and we have

\[
\sigma_R f(x) = \frac{1}{R} \int_0^R S_t f(x) \, dt = F_R * f(x) \to f(x) \quad \text{(35)}
\]
Since the Fejer Kernel is a continuous function of moderate decrease, we can actually calculate its Fourier Transform. Let $f$ be any function with a Fourier Transform supported on $\mathbb{R}$ and $\hat{f}(\xi) \neq 0$

\[
F_R * f = \frac{1}{R} \int_0^R D_t(x) \, dt * f
\]  
(36)

\[
= \frac{1}{R} \int_0^R (D_t(x) * f) \, dt
\]  
(37)

\[
= \frac{1}{R} \int_0^R \left( \int_{-t}^t \hat{f}(\xi) e^{2\pi i x \xi} \, d\xi \right) \, dt
\]  
(38)

\[
= \int_{-R}^R \hat{f}(\xi) e^{2\pi i x \xi} \left( \int_{|\xi|}^R \frac{1}{R} \, dt \right) \, d\xi
\]  
(39)
Fourier Transform of the Fejer Kernel

\[ F_R * f = \int_{-R}^{R} \hat{f}(\xi) e^{2\pi i x \xi} \left( 1 - \frac{|\xi|}{R} \right) d\xi \]

\[ = \int_{-\infty}^{\infty} \hat{f}(\xi) \left( 1 - \frac{|\xi|}{R} \right) \chi_{[-R,R]}(\xi) e^{2\pi i x \xi} d\xi \]

\[ = \left( \hat{f}(\xi) \left( 1 - \frac{|\xi|}{R} \right) \chi_{[-R,R]}(\xi) \right)^\sim \]

Now we can take the transform,

\[ \widehat{F_R * f} = \hat{f}(\xi) \left( 1 - \frac{|\xi|}{R} \right) \chi_{[-R,R]}(\xi) \]

\[ \widehat{F_R(\xi)} \hat{f}(\xi) = \hat{f}(\xi) \left( 1 - \frac{|\xi|}{R} \right) \chi_{[-R,R]}(\xi) \]

\[ \widehat{F_R(\xi)} = \left( 1 - \frac{|\xi|}{R} \right) \chi_{[-R,R]}(\xi) \]
Consider a real valued function $f \in L^2(\mathbb{R})$. Let $F(z)$ be twice the analytic extension of $f$ to the upper half plane $R_+^2 = \{z = x + iy : y > 0\}$. Then $F(z)$ is given explicitly by the well known Cauchy Integral Formula:

$$F(z) = \frac{1}{\pi i} \int_R \frac{f(t)}{t-z} \, dt$$ (46)

If we separate $F(z)$ into its real and imaginary parts, we get

$$F(z) = \frac{1}{\pi i} \int_R \frac{f(t)}{(t-x) - iy} \, dt$$ (47)
Connection with Complex Variables

\[ F(z) = \frac{1}{\pi i} \int_R \frac{f(t)((t-x)+iy)}{(t-x)^2+y^2} \, dt \]  

\[ = \frac{i}{\pi} \int_R \frac{f(t)(x-t)}{(t-x)^2+y^2} \, dt + \frac{1}{\pi} \int_R \frac{f(t)y}{(t-x)^2+y^2} \, dt \]  

\[ = f \ast P_y(x) + i(f \ast Q_y(x)) \]  

\[ = u + iv \]  

The function \( u \) is called the harmonic extension of \( f \) to the upper half plane, while \( v \) is called the harmonic conjugate of \( u \).
Theorem

\( Q_y \notin L^1(R) \), but \( Q_y \in L^2(R) \)

Proof

\[
\int_R \left| \frac{x}{\pi(x^2 + y^2)} \right| dx = \int_0^\infty \frac{-x}{\pi(x^2 + y^2)} dx + \int_0^\infty \frac{x}{\pi(x^2 + y^2)} dx
\]

\[
= -\int_\infty^{y^2} \frac{1}{2\pi} \frac{1}{u} du + \int_0^\infty \frac{1}{2\pi} \frac{1}{u} du
\]

\[
= \frac{1}{2\pi} \int_0^{\infty} \frac{1}{u} du + \int_0^{\infty} \frac{1}{2\pi} \frac{1}{u} du
\]

\[
= \frac{1}{\pi} \int_0^{\infty} \frac{1}{u} du
\]

\[
= \frac{1}{\pi} \ln |u|_y^\infty
\]

\[
= \infty
\]

So \( Q_y(x) \notin L_1(R) \)
The conjugate Poisson Kernel is not in $L^1(R)$ and is not a function of moderate decrease, so we define its Fourier transform as a principal value:

$$Q_y(\xi) = \lim_{R \to \infty} \int_{-R}^{R} e^{-2\pi i \xi x} Q_y(x) \, dx$$

$$= \lim_{R \to \infty} \int_{-R}^{R} e^{-2\pi i \xi x} \frac{x}{\pi (x^2 + y^2)} \, dx$$

Now, consider the function $f(z) = e^{-2\pi i \xi z} \frac{z}{\pi (z^2 + y^2)}$ and apply the residue theorem for a well-chosen contour in the complex plane.
For $\xi < 0$ we choose the semi circle in the upper half plane and for $\xi > 0$ we choose the semi circle in the lower half plane. Now, let's find the residues of $f$ at $iy$ and $-iy$.

$$Res(f(z), iy) = \frac{e^{-2\pi i \xi (iy)}}{2\pi}$$

$$= \frac{e^{2\pi i \xi y}}{2\pi} \quad \text{for} \quad \xi < 0 \quad (56)$$

$$Res(f(z), -iy) = \frac{e^{-2\pi i \xi (-iy)}}{2\pi}$$

$$= \frac{e^{-2\pi i \xi y}}{2\pi} \quad \text{for} \quad \xi > 0 \quad (58)$$
Connection with complex variables

Therefore

\[ Q_y(\xi) = -isgn(\xi)e^{-2\pi y|\xi|} \]  

\( (59) \)

**Definition**

The Hilbert Transform \( H \) is defined by:

\[ Hf(x) = \frac{1}{\pi} \lim_{\epsilon \to 0} \int_{|x-y| > \epsilon} \frac{f(y)}{x-y} \, dy \quad \text{for} \quad f \in L^2(\mathbb{R}) \]

**Definition**

\[ \hat{H}f(\xi) = -isgn(\xi)\hat{f}(\xi) \]

We can see that the Hilbert transform can be written as the principal value of the convolution of \( f \) and \( \frac{1}{\pi x} \). It is the response to \( f \) of a linear time invariant filter (Hilbert Transformer) having impulse response \( \frac{1}{\pi x} \).
The Hilbert transform finds a harmonic conjugate $y(t)$ for a real function $x(t)$ so that $z(t) = x(t) + iy(t)$ can be analytically extended from $R$ to the upper half plane. In signal processing, the HT can be interpreted as a way to represent a narrow band signal in terms of amplitude and frequency modulation. In the Fourier side, we can think of the HT as a phase shifter by 90 degrees. If we take the limit of $Q_y(\xi)$ as $y$ goes to zero, we get

$$Q_y(\xi) \rightarrow -isgn(\xi) \quad (60)$$

Then as $y \rightarrow 0$, $Q_y(x) \ast f$ approaches the Hilbert transform $Hf$ in $L^2(\mathbb{R})$. 
Consider the initial value problem for the heat equation:
\[ u_t - u_{xx} = 0 \text{ in } \mathbb{R} \times (0, \infty) \text{ with } u = g \text{ on } \mathbb{R} \times (t = 0) \]
If we take the Fourier transform of \( u \) in \( x \) we get: \( \hat{u}_t + y^2 \hat{u} = 0 \) for \( t > 0 \) and \( \hat{u} = \hat{g} \) for \( t = 0 \) Consequently, \( \hat{u} = e^{-ty^2} \hat{g} \) and \( u = e^{-ty^2} \hat{g} \) therefore \( u = \frac{g \ast \hat{F}}{(2\pi)^{\frac{1}{2}}} \) where \( \hat{F} = e^{-ty^2} \). But then

\[
F = (e^{-ty^2})
\]

\[
= \frac{1}{(2\pi)^{\frac{1}{2}}} \int_{\mathbb{R}} e^{i xy - ty^2} dy
\]

\[
= \frac{1}{2t} e^{-\frac{x^2}{4t}}
\]

Therefore,

\[
u(x, t) = \frac{1}{(4\pi t)^{\frac{1}{2}}} \int_{\mathbb{R}} e^{-|x-y|^2} g(y) dy \quad x \in \mathbb{R}, \quad t > 0
\]
References