A Bowl of Kernels

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Introduction

When we studied Fourier series we improved convergence by convolving with the Fejer and Poisson kernels on $\mathbb{T}$. The analogous Fejer and Poisson kernels on the real line help us improve convergence of the Fourier integral. In this project we define the Fejer kernel,the Poisson kernel, the heat kernel, the Dirichlet kernel, and the conjugate Poisson kernel. We study some of their properties, applications, and connections with complex variables.

Kernels

Fejer Kernel on R

$$F_R(x) = R\left(\frac{\sin \pi R x}{\pi R x}\right)^2 \qquad R > 0 \tag{1}$$

The Poisson kernel on $\mathbb{R}$

$$P_y(x) = \frac{y}{\pi(x^2 + y^2)} \qquad y > 0 \tag{2}$$

The Heat Kernel on $\mathbb{R}$

$$H_t(x) = \frac{e^{\frac{-|x|^2}{4t}}}{\sqrt{4\pi t}} \qquad t > 0 \tag{3}$$

Kernels

The Dirichlet kernel on $\mathbb{R}$

$$D_R(x) = \int_{-R}^{R} e^{2\pi i x \xi} d\xi = \frac{\sin(2\pi R x)}{\pi x} \qquad R > 0 \tag{4}$$

The Conjugate Poisson kernel on $\mathbb{R}$

$$Q_y(x) = \frac{x}{\pi(x^2 + y^2)} \qquad y > 0 \tag{5}$$

Kernels

Poisson Kernel and Conjugate Poisson Kernel solve the Laplace equation. $\Delta P = (\frac{\partial^2}{\partial x^2})P + (\frac{\partial^2}{\partial y^2})P = 0$,
$\Delta Q = (\frac{\partial^2}{\partial x^2})Q + (\frac{\partial^2}{\partial y^2})Q = 0$ for (x, y) in the upper half-plane $(x \in R, y > 0)$.
Proof(Poisson Kernel):

$$\frac{\partial P}{\partial x} = \frac{y}{\pi}(-1)(x^2 + y^2)^{-2}2x$$

$$\frac{\partial P}{\partial x^2} = \frac{8x^2y}{\pi(x^2 + y^2)^3} - \frac{2y}{\pi(x^2 + y^2)^2}$$

$$\frac{\partial P}{\partial y} = \frac{x^2 - y^2}{\pi(x^2 + y^2)^2}$$

$$\frac{\partial P}{\partial y^2} = \frac{-6yx^2 + 2y^3}{\pi(x^2 + y^2)^3}$$

Kernels

$$\begin{aligned}\Delta P &= \frac{8x^2y}{\pi(x^2+y^2)^3} - \frac{2y}{\pi(x^2+y^2)^2} + \frac{-6yx^2+2y^3}{\pi(x^2+y^2)^3} \\ &= \frac{2x^2y+2y^3-2y(x^2+y^2)}{\pi(x^2+y^2)^3} \\ &= 0\end{aligned}$$

Theorem

Heat Kernel is a solution of the heat equation on the line.
$(\frac{\partial}{\partial t})H = (\frac{\partial^2}{\partial x^2})H$

Approximation of The Identity

In class we learned that an Approximation of the Identity on **R** is a family $\{K_t\}_{t\in\Lambda}$ where the following holds:

1. The functions $\{K_t\}_{t\in\Lambda}$ have mean value 1: $\int_{-\infty}^{\infty} K_t(x)dx = 1$
2. The functions are uniformly integrable in t: There is a constant C $\geqslant 0$ such that $\int_{-\infty}^{\infty} |K_t(x)|dx \leqslant C$ for all $t \in \Lambda$
3. Concentration of mass at the origin: For each $\delta > 0 \quad \lim_{t\to t_0} \int_{|x|\geqslant\delta} |K_t(x)|dx = 0$

Fejer Kernel is an Approximation to the Identity

Recall the Fejer Kernel is defined as,

$$F_R(x) = R\left(\frac{\sin \pi Rx}{\pi Rx}\right)^2 \qquad R > 0 \tag{6}$$

Mean value 1 :

$$\int_R R\left(\frac{\sin \pi Rx}{\pi Rx}\right)^2 dx = 1 \tag{7}$$

Let $u = \pi Rx$, $\qquad dx = \frac{du}{\pi R}$ $\qquad$ Then we have,

$$\frac{1}{\pi}\int_R \left(\frac{\sin u}{u}\right)^2 du = \frac{1}{\pi}\int_R \frac{1-\cos^2 u}{u^2} du \tag{8}$$

$$= \frac{1}{\pi}\int_R \frac{(1-\cos u)(1+\cos u)}{u^2} du \tag{9}$$

Fejer Kernel

Now we can use integration by parts with $u' = 1 + \cos u$ and $dv = \frac{1-\cos u}{\pi u^2} du'$
Then we can use the well known identity, $\int_R \frac{1-\cos x}{x^2} dx = \pi$ and we have,

$$= (1 + \cos u) + \int_R \sin u' du' = (1 + \cos u) - \cos u = 1 \qquad (10)$$

Uniformly Integrable in t:
There is a constant C $\geqslant 0$ such that $\int_{-\infty}^{\infty} |R\left(\frac{\sin \pi Rx}{\pi Rx}\right)^2| dx \leqslant C$ for all $R > 0$
But by 1, we know $\int_R R\left(\frac{\sin \pi Rx}{\pi Rx}\right)^2 dx = 1$ and also $R\left(\frac{\sin \pi Rx}{\pi Rx}\right)^2 \geqslant 0$
So just let C=1.

Fejer Kernel

Concentration of mass at the origin: For each $\delta > 0 \quad \lim_{R\to\infty} \int_{|x|\geqslant\delta} |R\left(\frac{\sin \pi Rx}{\pi Rx}\right)^2| dx = 0.$

Let $\delta > 0$,

We know

$R\left(\frac{\sin \pi Rx}{\pi Rx}\right)^2 \leqslant R\frac{1}{(\pi Rx)^2} = \frac{1}{R(\pi x)^2}$

So we have,

$$\int_{|x|\geqslant\delta} R\left(\frac{\sin \pi Rx}{\pi Rx}\right)^2 \leqslant 2\int_{\delta}^{\infty} \frac{1}{R(\pi x)^2} \tag{11}$$

$$= 2 \lim_{L\to\infty} +\frac{1}{R(\pi)^2 L} - \frac{1}{R(\pi)^2 \delta} \tag{12}$$

$$= \frac{2}{R(\pi)^2 \delta} \tag{13}$$

Now we take the limit in R,

$$\lim_{R\to\infty} \frac{2}{R(\pi)^2 \delta} = 0 \tag{14}$$

Continuous Functions of Moderate Decrease

Definition

A function f is said to be a continuous function of moderate decrease if f is continuous and if there are constants A and $\epsilon > 0$ such that
$|f(x)| \leqslant \frac{A}{(1+|x|^{1+\epsilon})}$ for all $x \in R$

Theorem

Neither the Poisson kernel nor the Fejer kernel belong to $S(R)$. For each $R > 0$ and each $y > 0$ the Fejer kernel F_R and the Poisson kernel P_y are continuous functions of moderate decrease with $\epsilon = 1$. The conjugate Poisson kernel Q_y is not a continuous function of moderate decrease.

Proof(For Poisson Kernel) $P_y(x) = \frac{y}{\pi(x^2+y^2)}$ for $y > 0$
If $0 < y < 1$,

$$|P_y(x)| = |\frac{y}{\pi y^2((\frac{x}{y})^2 + 1)}|$$
$$= \frac{1}{(\pi y)(1 + \frac{x^2}{y^2})}$$

If $0 < y < 1$ then

$$\begin{aligned} y^2 &< 1 \\ \frac{1}{y^2} &> 1 \\ \frac{x^2}{y^2} &> x^2 \\ \frac{x^2}{y^2} + 1 &> x^2 + 1 \end{aligned}$$

Then

$$\begin{aligned} |P_y(x)| &= \frac{1}{(\pi y)(1 + \frac{x^2}{y^2})} \\ &< \frac{1}{\pi y(x^2 + 1)} \end{aligned}$$

If $y \geqslant 1$

$$y \geqslant 1$$
$$y^2 \geqslant 1$$
$$x^2 + y^2 gleqx^2 + 1$$

Then

$$|P_y(x)| = |\frac{y}{\pi y^2((\frac{x}{y})^2 + 1)}|$$
$$\leqslant \frac{y}{\pi(x^2 + 1)}$$

Hence,

$$A_y = \begin{cases} \frac{1}{\pi y} & \text{if } 0 < y < 1, \\ \frac{y}{\pi} & \text{if } y \geqslant 1 \end{cases}$$

Kernels

Theorem

Let $S_R f$ denote the partial Fourier integral of f, defined by

$$S_R f(x) = \int_{-R}^{R} \hat{f}(\xi) e^{2\pi i x \xi} d\xi \tag{15}$$

Then $S_R f(x) = D_R * f$ and $F_R(x) = \frac{1}{R}\int_0^R D_t(x)dt$.

Proof:

$$D_R * f = \int_R D_R(x-y) f(y) dy \tag{16}$$

$$= \int_R \int_{-R}^{R} e^{2\pi(x-y)\xi} d\xi f(y) dy \tag{17}$$

$$= \int_{-R}^{R} \int_R e^{-2\pi i y \xi} f(y) dy e^{2\pi i x \xi} d\xi \quad = \int_{-R}^{R} \hat{f(\xi)} e^{2\pi i x \xi} d\xi \tag{18}$$

Kernels

Also, $F_R(x) = \frac{1}{R}\int_0^R D_t(x)dt$ because,

$$\frac{1}{R}\int_0^R D_t(x)dt = \frac{1}{R}\int_0^R \frac{\sin(2\pi tx)}{\pi x}dt \tag{20}$$

$$= \frac{1}{R}\left(\frac{-1}{\pi x}\frac{\cos(2\pi tx)}{2\pi x}\right)\Bigg|_0^R \tag{21}$$

$$= \frac{1}{R2\pi^2x^2}\left(\cos(0) - \cos(2\pi Rx)\right) \tag{22}$$

$$= \frac{(\sin(\pi Rx)^2}{R\pi^2x^2} = F_R(x) \tag{23}$$

Integral Cesaro Means

We have the Fejer Kernel can be written as the integral mean of the Dirichlet kernel, and the Fejer kernel define an approximation to the identity on $\mathbb{R}$ as $R \to \infty$. Therefore, the integral Cesaro means of a function of moderate decrease converge to f as $R \to \infty$

Theorem

$$\sigma_R f(x) = \frac{1}{R}\int_0^R S_t f(x)dt = F_R * f(x) \to f(x) \tag{24}$$

The convergence is both uniform and in L^p

Integral Cesaro Means

We will prove for p=1.We will show,

$$\lim_{R\to\infty} ||F_R * f - f||_1 = 0 \tag{25}$$

$$|F_R(x) * f(x) - f(x)| = \left|\int_R f(x-y)F_R(y)dy - f(x)\right| \tag{26}$$

$$= \left|\int_R (f(x-y) - f(x))F_R(y)dy\right| \tag{27}$$

$$\leqslant \int_R |(f(x-y) - f(x))|\,|F_R(y)|\,dy \tag{28}$$

$$||F_R * f - f||_1 \leqslant \int_R\int_R |(f(x-y) - f(x))|\,|F_R(y)|\,dydx \tag{29}$$

$$\tag{30}$$

Integral Cesaro Means

By Fubini,

$$||F_R * f - f||_1 \leqslant \int_R ||(f(x-y) - f(x))||_1 \, |F_R(y)| \, dy \tag{31}$$

$$= \int_{|y|>\delta} ||(f(x-y) - f(x))||_1 \, |F_R(y)| \, dy \tag{32}$$

$$+ \int_{|y|<\delta} ||(f(x-y) - f(x))||_1 \, |F_R(y)| \, dy \tag{33}$$

$$= I_1 + I_2 \tag{34}$$

Notice, $I_2 = \int_{|y|<\delta} ||(f(x-y) - f(x))||_1 \, |F_R(y)| \, dy$ can be made arbitrarily small by continuity of f.

Integral Cesaro Means

Also, $I_1 = \int_{|y|>\delta} ||(f(x-y) - f(x))|\,|_1 \, |F_R(y)| \, dy$ can be made arbitrarily small since F_R is an approximation to the identity. This concludes the proof, and we have

$$\sigma_R f(x) = \frac{1}{R} \int_0^R S_t f(x) dt = F_R * f(x) \to f(x) \tag{35}$$

Fourier Transform of the Fejer Kernel

Since the Fejer Kernel is a continuous function of moderate decrease, we can actualy calculate its Fourier Transform.
Let f be any function with a Fourier Transform supported on $\mathbb{R}$ and $\hat{f}(\xi) \neq 0$

$$F_R * f = \frac{1}{R}\int_0^R D_t(x)dt * f \tag{36}$$

$$= \frac{1}{R}\int_0^R (D_t(x) * f)dt \tag{37}$$

$$= \frac{1}{R}\int_0^R \left(\int_{-t}^{t} \hat{f}(\xi)e^{2\pi i x\xi}d\xi\right) dt \tag{38}$$

$$= \int_{-R}^{R} \hat{f}(\xi)e^{2\pi i x\xi}\left(\int_{|\xi|}^{R} \frac{1}{R}dt\right) d\xi \tag{39}$$

Fourier Transform of the Fejer Kernel

$$F_R * f = \int_{-R}^{R} \hat{f}(\xi) e^{2\pi i x \xi} \left(1 - \frac{|\xi|}{R}\right) d\xi \tag{40}$$

$$= \int_{-\infty}^{\infty} \hat{f}(\xi) \left(1 - \frac{|\xi|}{R}\right) \chi_{[-R,R]}(\xi) e^{2\pi i x \xi} d\xi \tag{41}$$

$$= \left(\hat{f}(\xi) \left(1 - \frac{|\xi|}{R}\right) \chi_{[-R,R]}(\xi)\right)^{\vee} \tag{42}$$

Now we can take the transform,

$$\widehat{F_R * f} = \hat{f}(\xi) \left(1 - \frac{|\xi|}{R}\right) \chi_{[-R,R]}(\xi) \tag{43}$$

$$\widehat{F_R(\xi)} \hat{f(\xi)} = \hat{f}(\xi) \left(1 - \frac{|\xi|}{R}\right) \chi_{[-R,R]}(\xi) \tag{44}$$

$$\widehat{F_R(\xi)} = \left(1 - \frac{|\xi|}{R}\right) \chi_{[-R,R]}(\xi) \tag{45}$$

Connection with Complex Variables

Consider a real valued function $f \in L^2(\mathbb{R})$. Let $F(z)$ be twice the analytic extension of f to the upper half plane $R^2_+ = \{z = x + iy : y > 0\}$ Then $F(z)$ is given explicitly by the well known Cauchy Integral Formula:

$$F(z) = \frac{1}{\pi i} \int_R \frac{f(t)}{t - z} dt \tag{46}$$

If we separate $F(z)$ into its real and imaginary parts, we get

$$F(z) = \frac{1}{\pi i} \int_R \frac{f(t)}{(t - x) - iy} dt \tag{47}$$

Connection with Complex Variables

$$F(z) = \frac{1}{\pi i}\int_R \frac{f(t)((t-x)+iy)}{(t-x)^2+y^2}dt \tag{48}$$

$$= \frac{i}{\pi}\int_R \frac{f(t)(x-t)}{(t-x)^2+y^2}dt + \frac{1}{\pi}\int_R \frac{f(t)y}{(t-x)^2+y^2}dt \tag{49}$$

$$= f * P_y(x) + i(f * Q_y(x)) \tag{50}$$

$$= u + iv \tag{51}$$

The function u is called the harmonic extension of f to the upper half plane , while v is called the harmonic conjugate of u.

Theorem

$Q_y \notin L^1(R)$, *but* $Q_y \in L^2(R)$

Proof

$$\begin{aligned}
\int_R \left|\frac{x}{\pi(x^2+y^2)}\right| dx &= \int_{-\infty}^{0} \frac{-x}{\pi(x^2+y^2)} dx + \int_0^{\infty} \frac{x}{\pi(x^2+y^2)} \\
&= -\int_{\infty}^{y^2} \frac{1}{2\pi}\frac{1}{u} du + \int_{y^2}^{\infty} \frac{1}{2\pi}\frac{1}{u} du \\
&= \frac{1}{2\pi}\int_{y^2}^{\infty} \frac{1}{u} du + \int_{y^2}^{\infty} \frac{1}{2\pi}\frac{1}{u} du \\
&= \frac{1}{\pi}\int_{y^2}^{\infty} \frac{1}{u} du \\
&= \frac{1}{\pi} ln|u|_{y^2}^{\infty} \\
&= \infty
\end{aligned}$$

So $Q_y(x) \notin L_1(R)$

Connection with Complex Variables

The conjugate Poisson Kernel is not in $L^1(R)$ and is not a function of moderate decrease, so we define its Fourier transform as a principal value:

$$\hat{Q_y}(\xi) = \lim_{R\to\infty} \int_{-R}^{R} e^{-2\pi i\xi x} Q_y(x)\,dx \tag{52}$$

$$= \lim_{R\to\infty} \int_{-R}^{R} e^{-2\pi i\xi x} \frac{x}{\pi(x^2+y^2)}\,dx \tag{53}$$

$$\tag{54}$$

Now, consider the function $f(z) = e^{-2\pi i\xi z}\frac{z}{\pi(z^2+y^2)}$ and apply residue theorem for a well chosen contour in the complex plane.

connection to complex variables

For $\xi < 0$ we choose the semi circle in the upper half plane and for $\xi > 0$ we choose the semi circle in the lower half plane. Now, lets find the residues of f at iy and $-iy$.

$$Res(f(z), iy) = \frac{e^{-2\pi i\xi(iy)}}{2\pi} \tag{55}$$

$$= \frac{e^{2\pi i\xi y}}{2\pi} \qquad for \qquad \xi < o \tag{56}$$

$$Res(f(z), -iy) = \frac{e^{-2\pi i\xi(-iy)}}{2\pi} \tag{57}$$

$$= \frac{e^{-2\pi i\xi y}}{2\pi} \qquad for \qquad \xi > 0 \tag{58}$$

Connection with complex variables

Therefore

$$\hat{Q_y}(\xi) = -isgn(\xi)e^{-2\pi y|\xi|} \tag{59}$$

Definition

The Hilbert Transform H is defined by :
$Hf(x) = \frac{1}{\pi}\lim_{\epsilon\to 0}\int_{|x-y|>\epsilon}\frac{f(y)}{x-y}dy \qquad for \qquad f \in L^2(\mathbb{R})$

Definition

$\hat{H}f(\xi) = -isgn(\xi)\hat{f}(\xi)$

We can see that the Hilbert transform can be written as the principal value of the convolution of f and $\frac{1}{\pi x}$. It is the response to f of a linear time invariant filter(Hilbert Transformer) having impulse response $\frac{1}{\pi x}$

Connection with Complex Variables

The Hilbert transform find a harmonic conjugate $y(t)$ for a real function $x(t)$ so that $z(t) = x(t) + iy(t)$ can be analytically extended from R to the upper half plane.In signal processing, the HT can be interpreted as a way to represent a narrow band signal in terms of amplitude and frequency modulation.In the Fourier side, we can think of the HT as a phase shifter by 90 degrees. If we take the limit of $\hat{Q_y}(\xi)$ as y goes to zero, we get

$$\hat{Q_y}(\xi) \to -isgn(\xi) \tag{60}$$

Then as y $\to$ 0, $Q_y(x) * f$ approaches the Hilbert transform Hf in $L^2(\mathbb{R})$

Applications

Consider the initial value problem for the heat equation:
$u_t - u_{xx} = 0$ in $\mathbb{R}x(0,\infty)$ with $u = g$ on $\mathbb{R}x(t=0)$
If we take the Fourier transform of u in x we get: $\hat{u}_t + y^2\hat{u} = 0$ for $t > 0$ and $\hat{u} = \hat{g}$ for $t = 0$ Consequently, $\hat{u} = e^{-ty^2}\hat{g}$ and $u = \widehat{e^{-ty^2}\hat{g}}$ therefore $u = \frac{g*F}{(2\pi)^{\frac{1}{2}}}$ where $\hat{F} = e^{-ty^2}$. But then

$$F = \widehat{(e^{-ty^2})} \tag{61}$$

$$= \frac{1}{(2\pi)^{\frac{1}{2}}} \int_{\mathbb{R}} e^{ixy - ty^2} dy \tag{62}$$

$$= \frac{1}{2t} e^{\frac{-x^2}{4t}} \tag{63}$$

Therefore,

$$u(x,t) = \frac{1}{(4\pi t)^{\frac{1}{2}}} \int_{\mathbb{R}} e^{-|x-y|^2} g(y) dy \qquad x \in \mathbb{R}, \qquad t > 0 \tag{64}$$

References

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