Multiple Fourier Series

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Agenda

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Fourier series in one dimension

- A periodic function $f(x)$ with a period of 2π and for which $\int_0^{2\pi} f(x)^2 dx$ is finite has a Fourier series expansion

$$f(x) \sim \frac{1}{2}a_0 + \sum_{n=1}^{\infty} [a_n \; cos \; nx + b_n \; sin \; nx]$$

  and, this fourier series converges to $f(x)$ in the mean [Weinberger, 1965].

- If $f(x)$ is continuously differentiable, its Fourier series converges uniformly.

Periodic Functions

- Consider a function $f(x_1, x_2)$ (p_1, p_2)-periodic in variables x_1 and x_2 [Osgood, 2007]

$$f(x_1 + n_1 p_1, x_2 + n_2 p_2) = f(x_1, x_2) \;\; \forall \;\; x_1, x_2 \in \mathcal{R}; n_1, n_2 \in \mathcal{Z}.$$

- Assuming p_1 and p_2 to be 1, the new condition is

$$f(x_1 + n_1, x_2 + n_2) = f(x_1, x_2) \;\; \forall \;\; x_1, x_2 \in [0, 1]^2.$$

- If we use vector notation, and write $\mathbf{x}$ for (x_1, x_2), and $\mathbf{n}$ for pairs (n_1, n_2) of integers, then we can write the condition as

$$f(\mathbf{x+n}) = f(\mathbf{x}) \;\; \forall \;\; x \in [0, 1]^2, n \in \mathcal{N}.$$

- In $\mathbf{d}$ dimensions, we have $\mathbf{x} = (x_1, x_2, ...x_d)$ and $\mathbf{n} = (n_1, n_2, ...n_d)$. and so the vector notation becomes

$$f(\mathbf{x+n}) = f(\mathbf{x}) \;\; \forall \;\; x \in [0, 1]^d, n \in \mathcal{N}.$$

Complex Exponentials

- In 2-D, the building blocks for periodic function $f(x_1, x_2)$ are the product of complex exponentials in one variable. The general higher harmonic is of the form

$$e^{2\pi i n_1 x_1} \; e^{2\pi i n_2 x_2},$$

and we can imagine writing the Fourier series expansion as

$$\sum_{n_1, n_2} c_{n_1, n_2} e^{2\pi i n_1 x_1} e^{2\pi i n_2 x_2},$$

with an equivalent vector notation using $\mathbf{n} = (n_1, n_2)$.

$$\sum_{\mathbf{n} \in \mathcal{Z}^2} c_{\mathbf{n}} \; e^{2\pi i n_1 x_1} \; e^{2\pi i n_2 x_2}.$$

- So the Fourier series expansion in **2-D** looks like

$$\sum_{\mathbf{n} \in \mathcal{Z}^2} c_{\mathbf{n}} \; e^{2\pi i \mathbf{n}.\mathbf{x}} \; .$$

Complex Exponentials (contd.)

- Similarly, in **d**-D, the corresponding complex exponential is

$$e^{2\pi i n_1 x_1} \; e^{2\pi i n_2 x_2} \; .... \; e^{2\pi i n_d x_d},$$

  and we can imagine writing the Fourier series expansion as

$$\sum_{n_1, n_2, .., n_d} c_{n_1, n_2, .. n_d} \; e^{2\pi i n_1 x_1} \; e^{2\pi i n_2 x_2} ... \; e^{2\pi i n_d x_d}.$$

  with an equivalent vector notation using $\mathbf{n} = (n_1, n_2, ..., n_d)$.

$$\sum_{\mathbf{n} \in \mathcal{Z}^d} c_{\mathbf{n}} \; e^{2\pi i n_1 x_1} \; e^{2\pi i n_2 x_2} \; e^{2\pi i n_d x_d} \; .$$

- So the Fourier series expansion in **d-D** looks like

$$\sum_{\mathbf{n} \in \mathcal{Z}^d} c_{\mathbf{n}} \; e^{2\pi i \mathbf{n.x}} \; .$$

Vector Notation Summarized

- The Fourier series expansion in **d-D** is approximated as

$$f(\mathbf{x}) = \sum_{\mathbf{n}\in\mathcal{Z}^d} c_{\mathbf{n}} \ e^{2\pi i \mathbf{n}.\mathbf{x}} \ ,$$

where $\mathbf{x} = [x_1, x_2, \dots, x_d] \in [0,1]^d$, and $\mathbf{n} = [n_1, n_2, \dots, n_d] \in \mathcal{Z}^d$.

- The Fourier co-efficients ($\hat{f} = c_{\mathbf{n}}$) can be defined by the integral

$$\begin{aligned}
\hat{f}(\mathbf{n}) &= \int_{[0,1]} \dots \int_{[0,1]} e^{-2\pi i n_1 x_1} e^{-2\pi i n_2 x_2} \dots e^{-2\pi i n_d x_d} f(x_1, x_2, ..x_d) dx_1..dx_d \\
&= \int_{[0,1]} \dots \int_{[0,1]} e^{-2\pi i (n_1 x_1 + n_2 x_2 + \dots + n_d x_d)} f(x_1, x_2, \dots, x_d) dx_1 dx_2 \dots dx_d \\
&= \int_{[0,1]^d} e^{-2\pi i \mathbf{n}.\mathbf{x}} f(\mathbf{x}) dx.
\end{aligned}$$

Fourier series in two dimensions

- Let $f(x,y)$ be a continuously differentiable periodic function with a period of 2π in both of the variables:

$$f(x+2\pi,y)=f(x,y+2\pi)=f(x,y).$$

- For each value of y, we can expand $f(x,y)$ in a uniformly convergent Fourier series

$$f(x,y)=\frac{1}{2}a_0(y)+\sum_{n=1}^{\infty}[a_n(y)\ cos\ nx+b_n(y)\ sin\ nx].$$

- The co-efficients

$$a_n(y)=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x,y)\ cos\ nx\ dx$$

$$b_n(y)=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x,y)\ sin\ nx\ dx$$

are continuously differentiable in y.

Fourier Series in two dimensions (contd.)

- Co-efficients can be expanded in uniformly convergent Fourier series

$$a_n(y) = \frac{1}{2}a_{n0} + \sum_{m=1}^{\infty}(a_{nm}\ cos\ my + b_{nm}\ sin\ my)$$

$$b_n(y) = \frac{1}{2}c_{n0} + \sum_{m=1}^{\infty}(c_{nm}\ cos\ my + d_{nm}\ sin\ my)$$

where

$$a_{nm} = \frac{1}{\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f(x,y)\ cos\ nx\ cos\ my\ dx\ dy$$

$$b_{nm} = \frac{1}{\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f(x,y)\ cos\ nx\ sin\ my\ dx\ dy$$

$$c_{nm} = \frac{1}{\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f(x,y)\ sin\ nx\ cos\ my\ dx\ dy$$

$$d_{nm} = \frac{1}{\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f(x,y)\ sin\ nx\ sin\ my\ dx\ dy.$$

Fourier Series in two dimensions (contd.)

- Putting the series for the coefficients into the series for f(x,y), we have

$$
\begin{aligned}
f(x,y) \sim \frac{1}{4}a_{00} &+ \frac{1}{2}\sum_{m=1}^{\infty}[a_{0m} cos\ my + b_{0m} sin\ my] \\
&+ \frac{1}{2}\sum_{n=1}^{\infty}[a_{n0} cos\ nx + c_{n0} sin\ nx] \\
&+ \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}[a_{nm}\ cos\ nx\ cos\ my + b_{nm}\ cos\ nx\ sin\ my \\
&\qquad + c_{nm}\ sin\ nx\ cos\ my + d_{nm}\ sin\ nx\ sin\ my]
\end{aligned}
$$

Proof of convergence of double Fourier series

- The Parseval equation gives

$$\int_{-\pi}^{\pi} f(x,y)^2 \; dx = \frac{\pi}{2} a_0(y)^2 + \pi \sum_{n=1}^{\infty} [a_n(y)^2 + b_n(y)^2].$$

- The series on right converges uniformly in y. Hence we may integrate with respect to y term by term:

$$\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f(x,y)^2 \; dx \; dy = \frac{\pi^2}{2} \int_{-\pi}^{\pi} a_0^2 \; dy + \pi^2 \sum_{n=1}^{\infty} \int_{-\pi}^{\pi} [a_n^2 + b_n^2] \; dy.$$

- We now apply the Parseval equation to the functions $a_n(y)$ and $b_n(y)$:

$$\int_{-\pi}^{\pi} a_n(y)^2 \; dy = \frac{\pi}{2} a_{n0}^2 + \sum_{m=1}^{\infty} (a_{nm}^2 + b_{nm}^2)$$

$$\int_{-\pi}^{\pi} b_n(y)^2 \; dy = \frac{\pi}{2} c_{n0}^2 + \sum_{m=1}^{\infty} (c_{nm}^2 + d_{nm}^2)$$

Proof of convergence of double Fourier series (contd.)

- Thus, we get the Parseval's equation for double Fourier series derived under the hypothesis that f(x,y) is continuously differentiable.

$$\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f(x,y)^2\,dx\,dy = \frac{\pi^2}{4}a_{00}^2 + \frac{\pi^2}{2}\sum_{m=1}^{\infty}(a_{0m}^2 + b_{0m}^2) + \frac{\pi^2}{2}\sum_{n=1}^{\infty}(a_{n0}^2 + c_{n0}^2) + \pi^2\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}(a_{nm}^2 + b_{nm}^2 + c_{nm}^2 + d_{nm}^2)$$

Proof of convergence of double Fourier series (contd.)

- Assuming f(x,y) is such that $\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f(x,y)^2 dxdy$ is finite implies that f(x,y) can be approximated in the mean by continuously differentiable functions. As such, Parseval equation remains valid for such functions.

- Additionally, we know that the functions $cos(nx)\ cos(my)$, $cos(nx)\ sin(my)$, $sin(nx)\ cos(my)$, and $sin(nx)\ sin(my)$ are orthogonal in the sense that

$$\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} cos(nx)cos(my)cos(kx)cos(ly)dxdy = 0 \ unless \ n = k, m = l,$$

$$\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} cos(nx)cos(my)cos(kx)sin(ly)dxdy = 0$$

and so forth.

Proof of convergence of double Fourier series (contd.)

- Therefore, we find that:

$$\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\Bigg[f(x,y)-\Bigg(\frac{1}{4}a_{00}+\frac{1}{2}\sum_{m=1}^{M}[a_{0m}cos(my)+b_{0m}sin(my)]+\frac{1}{2}\sum_{n=1}^{N}[a_{n0}cos(nx)+c_{n0}sin(nx)]$$

$$+\sum_{n=1}^{N}\sum_{m=1}^{M}[a_{nm}cos(nx)cos(my)]+[b_{nm}cos(nx)sin(my)]+[c_{nm}sin(nx)cos(my)]+[d_{nm}sin(nx)sin(my)]\Bigg)\Bigg]^2dxdy$$

$$=\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}f(x,y)^2\,dxdy-\Bigg(\frac{\pi^2}{4}a_{00}^2+\frac{\pi^2}{2}\sum_{m=1}^{M}[a_{0m}^2+b_{0m}^2]+\frac{\pi^2}{2}\sum_{n=1}^{N}[a_{n0}^2+c_{n0}^2]$$

$$+\pi^2\sum_{n=1}^{N}\sum_{m=1}^{M}[a_{nm}^2+b_{nm}^2+c_{nm}^2+d_{nm}^2]\Bigg)$$

- In the above expression we have used

$$\int_a^b[f(x)-\sum_1^N c_n\phi_n(x)]^2\rho(x)dx=\int_a^b f^2\rho dx-\sum_1^N c_n^2\int_a^b\phi_n^2\rho dx.$$

- By Parseval's equation, the R.H.S. approaches 0 as $N, M \to \infty$. So, Fourier series converges to f(x,y) in the mean as $N, M \to \infty$.

Proof of convergence of double Fourier series (contd.)

- Furthermore, it can be shown that if f(x,y) is continuous and continuously differentiable, and if the squares of its second partial derivatives have finite integrals, then the double fourier series converges absolutely and uniformly to f(x,y) as a double series.

Fourier series examples

Examples..

Laplace's Equation in a Cube

We consider the problem $\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$ where,

- $0 < x < \pi,\ 0 < y < \pi,\ 0 < z < \pi,$
- $u = 0$ for $x = 0,\ x = \pi,\ y = 0,\ y = \pi$, and $z = \pi$,
- $u(x, y, 0) = g(x, y)$

This problem arises in electrostatics when u is the potential whose value g is given on the face $z = 0$, while other faces are perfect conductors kept at zero potential.

u can also be interpreted as an equilibrium temperature distribution when the faces are kept at temperatures 0 and g, respectively.

Laplace's Equation in a Cube (contd.)

- Maximum principle holds for Laplace's equation in 3 as well as 2 dimensions, so this 3-D boundary value problem for Laplace's equation has at most one solution which varies continuously with boundary values.

- We use the method of separation of variables to solve this problem. Consider the product function

$$u = X(x)Y(y)Z(z)$$

that solves the Laplace's equation

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0.$$

- By this substitution, we get

$$\frac{\nabla^2 u}{u} = \frac{X''}{X} + \frac{Y''}{Y} + \frac{Z''}{Z} = 0 \Rightarrow \frac{X''}{X} + \frac{Y''}{Y} = -\frac{Z''}{Z} = C_1$$

Laplace's Equation in a Cube (contd.)

- Again,

$$\frac{X''}{X} = C_1 - \frac{Y''}{Y} = C_2.$$

So we have,

$$X'' - C_2 X = 0,$$
$$Y'' - (C_1 - C_2)Y = 0,$$
$$Z'' + C_1 Z = 0.$$

- The homogeneous boundary conditions give

$$X(0) = X(\pi) = 0,$$
$$Y(0) = Y(\pi) = 0,$$
$$Z(\pi) = 0.$$

Laplace's Equation in a Cube (contd.)

- We must have $C_2 = -n^2$, where n is a positive integer, with the corresponding eigen function

$$X = sin\ nx.$$

- For Y, we have $C_1 - C_2 = -m^2$, where m is another positive integer, and

$$Y = sin\ my.$$

- Then $C_1 = -m^2 - n^2$, so that Z is a multiple of

$$sinh\ \sqrt{m^2 + n^2}\ (\pi - z).$$

Laplace's Equation in a Cube (contd.)

- We seek a solution of the form

$$u(x, y, z) = \sum_{1}^{\infty} \sum_{1}^{\infty} \alpha_{nm} \, sinh \, \sqrt{m^2 + n^2} \, (\pi - z) \, sin \, nx \, sin \, my.$$

- Putting z = 0, we formally obtain

$$g(x, y) = u(x, y, 0) = \sum_{1}^{\infty} \sum_{1}^{\infty} \alpha_{nm} \, sinh \, \sqrt{m^2 + n^2} \, \pi \, sin \, nx \, sin \, my.$$

- Therefore from the double Fourier series expansion we have

$$\alpha_{nm} \, sinh \, \sqrt{m^2 + n^2} \, \pi = d_{mn} = \frac{4}{\pi^2} \int_0^{\pi} \int_0^{\pi} g(x, y) \, sin \, nx \, sin \, my \, dxdy.$$

- Then

$$u(x, y, z) = \sum_{1}^{\infty} \sum_{1}^{\infty} \frac{d_{nm}}{sinh\sqrt{n^2 + m^2}(\pi)} sinh\sqrt{n^2 + m^2}(\pi - z) \, sin \, nx \, sin \, my$$

3D Wave Equation in a Cube

We consider the problem

$$\frac{\partial^2 u}{\partial t^2} - c^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right) = 0$$

with initial conditions

$$\begin{aligned} u(0, y, z, t) &= u(\pi, y, z, t) = 0, \\ u(x, 0, z, t) &= u(x, \pi, z, t) = 0, \\ u(x, y, 0, t) &= u(x, y, \pi, t) = 0, \\ u(x, y, z, 0) &= f(x, y, z), \\ \frac{\partial u}{\partial t}(x, y, z, 0) &= g(x, y, z). \end{aligned}$$

Solution to this hyperbolic problem describes the propagation of sound waves from an initial disturbance in a cubical room...

3D Wave Equation in a Cube (contd.)

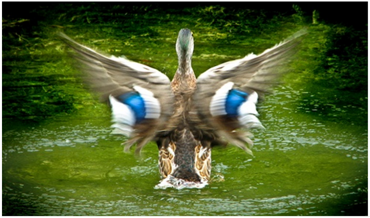
The formal solution to this problem is given by

$$u(x,y,z,t) = \sum\sum\sum[d_{lmn} cos\sqrt{l^2+m^2+n^2}ct + \tilde{d}_{lmn}\frac{sin\sqrt{l^2+m^2+n^2}ct}{\sqrt{l^2+m^2+n^2}c}]sin\ lx\ sin\ my\ sin\ nz,$$

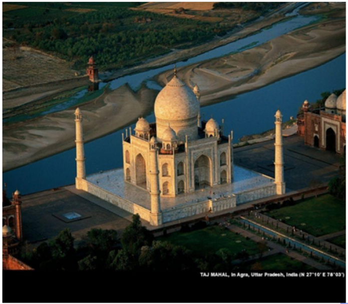
where

$$d_{lmn} = \frac{8}{\pi^3}\int_0^\pi\int_0^\pi\int_0^\pi f(x,y,z)\ sin\ lx\ sin\ my\ sin\ nz\ dxdydz,$$
$$\tilde{d}_{lmn} = \frac{8}{\pi^3}\int_0^\pi\int_0^\pi\int_0^\pi g(x,y,z)\ sin\ lx\ sin\ my\ sin\ nz\ dxdydz.$$

Symmetry in Nature



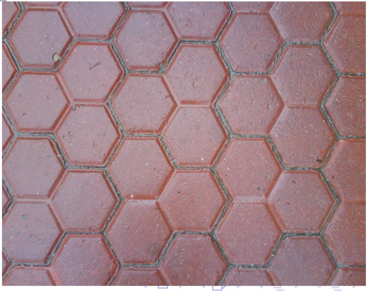
Symmetry in Architecture



Symmetry in Snowflakes

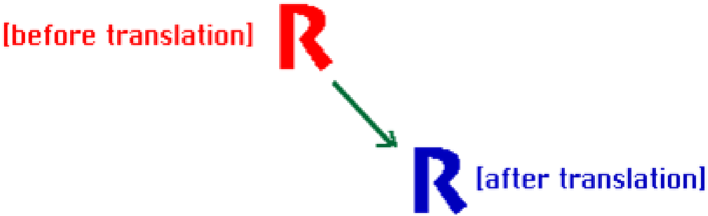
1-D Repetitive Patterns

2-D Repetitive Patterns



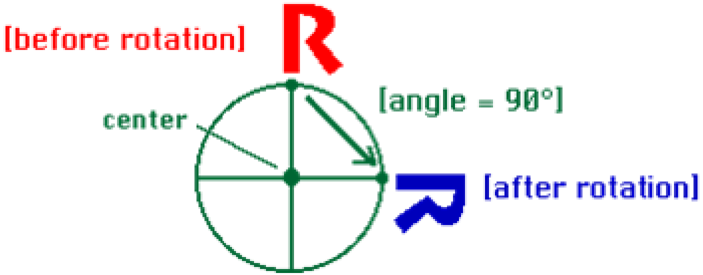
Isometeries of the Euclidean plane (Translation)

To translate an object means to move it without rotating or reflecting it. Every translation has a direction and a distance.



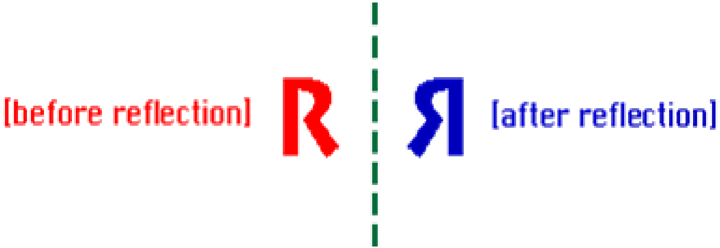
Isometeries of the Euclidean plane (Rotation)

To rotate an object means to turn it around. Every rotation has a center and an angle.



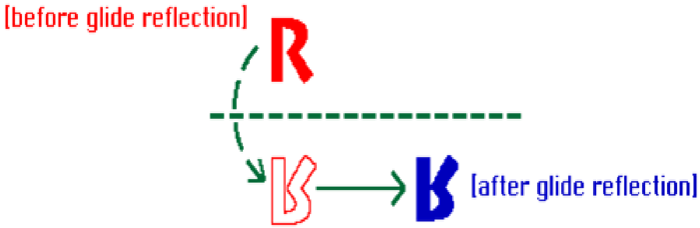
Isometeries of the Euclidean plane (Reflection)

To reflect an object means to produce its mirror image. Every reflection has a mirror line. A reflection of an "R" is a backwards "R".



Isometeries of the Euclidean plane (Glide Reflection)

A glide reflection combines a reflection with a translation along the direction of the mirror line. Glide reflections are the only type of symmetry that involve more than one step.



Seventeen 2-D patterns

Wallpaper Group	Finite Presentation
P111	$< x, y : xy = yx >$
P1m1	$< x, y, z : x^2 = y^2 = 1, xz = zx, yz = zy >$
P1g1	$< x, y : x^2 = y^2 >$
C1m1	$< x, y : x^2 = 1, xy^2 = y^2x >$
P211	$< t_1, t_2, t_3 : t_1^2 = t_2^2 = t_3^2 = (t_1t_2t_3)^2 = 1 >$
P2mm	$< r_1, r_2, r_3, r_4 : r_1^2 = r_2^2 = r_3^2 = r_4^2 =$ $(r_1r_2)^2 = (r_2r_3)^2 = (r_3r_4)^2 = (r_4r_1)^2 == 1 >$
P2mg	$< p, q, r : p^2 = q^2 = r^2 = 1, pq = rpr >$
P2gg	$< p, q, r : p^2 = q^2, r^2 = 1, rpr = q^{-1} >$
C2mm	$< p, q, r : p^2 = q^2 = r^2 = (pq)^2 = (prqr)^2 = 1 >$

Table: Wallpaper Group patterns and their Finite Presentations

Seventeen 2-D patterns

Wallpaper Group	Finite Presentation
P3	$< u, v, w : u^3 = v^3 = w^3 = uvw = 1 >$
P3m1	$< r, s : r^2 = s^3 = (rs^{-1}rs)^3 = 1 >$
P31m	$< p, q, r : p^2 = q^2 = r^2 = (pq)^3 = (qr)^3 = (rp)^3 = 1 >$
P4	$< a, b, c, d, e : a^2 = b^2 = c^2 = d^2 = abcd = 1,$
	$= e^5, e^{-1}de = a, e^{-2}de^2 = b, e^3de^{-3} = c >$
P4mm	$< p, q, r : p^2 = q^2 = r^2 =$
	$(pq)^4 = (qr)^2 = (rp)^4 = 1 >$
P4gm	$< a, b, c, d, e : a^2 = b^2 = c^2 = d^2$
	$= (ab)^2 >= (bc)^2 = (cd)^2 = (da)^2 = e^4 = 1$
P6	$< a, b : a^3 = b^2 = ab^6 = 1 >$
P6mm	$< p, q, r : p^2 = q^2 = r^2 = qr^3 = rp^2 = pq^6 = 1 >$

Table: Wallpaper Group patterns and their Finite Presentations

Discrete dynamical systems

The system given by the recurrence relations

$$\begin{aligned} x_{n+1} &= x_n - f(x_n, y_n) \\ y_{n+1} &= y_n - g(x_n, y_n) \end{aligned} \tag{1}$$

is called a discrete dynamical system.

Translational Symmetry

Suppose that the phase portrait has a period T along the x-axis. The phase portrait is invariant after the transformation $x^{'} = x + T$ and $y^{'} = y$. Substituting $x^{'}$ and $y^{'}$ into (1), we have

$$\begin{aligned} x^{'}_{n+1} &= x^{'}_{n} - f(x^{'}_{n} + T, y^{'}_{n}) \\ y^{'}_{n+1} &= y^{'}_{n} - g(x^{'}_{n} + T, y^{'}_{n}) \end{aligned} \tag{2}$$

For (1) and (2) to be identical, we must have

$$\begin{aligned} f(x + T, y) &= f(x, y) \\ g(x + T, y) &= g(x, y) \end{aligned} \tag{3}$$

Similarly, if the phase portrait has a period T^* along the y-axis, it can be shown that

$$\begin{aligned} f(x, y + T^*) &= f(x, y) \\ g(x, y + T^*) &= g(x, y) \end{aligned} \tag{4}$$

Reflection Symmetry

Suppose that the phase portrait has reflective symmetry about the x-axis. Let $x^{'} = x$ and $y^{'} = -y$. Then we have,

$$\begin{aligned} x^{'}_{n+1} &= x^{'}_{n} - f(x^{'}_{n}, -y^{'}_{n}) \\ y^{'}_{n+1} &= y^{'}_{n} + g(x^{'}_{n}, -y^{'}_{n}) \end{aligned} \tag{5}$$

From invariance of the transformation we obtain

$$\begin{aligned} f(x, -y) &= f(x, y) \\ g(x, -y) &= -g(x, y) \end{aligned} \tag{6}$$

Similarly, if the phase portrait has reflective symmetry about the y-axis, then

$$\begin{aligned} f(-x, y) &= -f(x, y) \\ g(-x, y) &= g(x, y) \end{aligned} \tag{7}$$

Glide Reflective Symmetry

Suppose that phase portrait has a period T along x-axis and a glide reflection in the same direction. Let $x^{'} = x + \frac{T}{2}$ and $y^{'} = -y$. Then,

$$\begin{aligned} x^{'}_{n+1} &= x^{'}_{n} - f(x^{'}_{n} + \frac{T}{2}, -y^{'}_{n}) \\ y^{'}_{n+1} &= y^{'}_{n} + g(x^{'}_{n} + \frac{T}{2}, -y^{'}_{n}) \end{aligned} \tag{8}$$

From invariance of the transformation we obtain

$$f(x + \frac{T}{2}, -y) = f(x, y),\ g(x + \frac{T}{2}, -y) = -g(x, y) \tag{9}$$

Similarly, if the phase portrait has a period T^* about the y-axis and a glide reflection in the same direction, then

$$\begin{aligned} f(-x, y + \frac{T^*}{2}) &= -f(x, y) \\ g(-x, y + \frac{T^*}{2}) &= g(x, y). \end{aligned} \tag{10}$$

Rotational Symmetry

Suppose that the phase portrait remains unchanged after a rotation of an angle θ counter clockwise. Let

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} cos\ \theta & -sin\ \theta \\ sin\ \theta & cos\ \theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = T_\theta \begin{bmatrix} x \\ y \end{bmatrix} \tag{11}$$

Substituting (11) into (1), we have

$$\begin{bmatrix} x'_{n+1} \\ y'_{n+1} \end{bmatrix} = \begin{bmatrix} x'_n \\ y'_n \end{bmatrix} - T_\theta \begin{bmatrix} f \\ g \end{bmatrix} \tag{12}$$

From (1) and (12), we have

$$\begin{aligned} f(x', y') &= cos\ \theta\ f(x, y) - sin\ \theta\ g(x, y) \\ g(x', y') &= sin\ \theta\ f(x, y) + cos\ \theta\ g(x, y) \end{aligned} \tag{13}$$

Eliminating g(x,y) from (13), we obtain

$$f(x'', y'') - 2cos\ \theta\ f(x', y') + f(x, y) = 0, \tag{14}$$

Dynamical systems with P6mm Symmetry

We chose the wallpaper group P6mm as an example and show how to construct dynamical system with this symmetry as shown in [Chung, 1993]. Patterns having other symmetries can be constructed in a similar fashion. P6mm has a 6-fold rotational symmetry. To obtain a phase portrait of (1) with P6mm symmetry we substitute $\theta = \frac{\pi}{3}$ into (14) and obtain

$$f(x^{''}, y^{''}) - f(x^{'}, y^{'}) + f(x, y) = 0. \tag{15}$$

To find the general solution of (15), we express $f(x, y)$ as a linear combination of the function $h(x^{(n)}, y^{(n)})(n = 0, 1, 2, 3, 4, 5)$ where h(x,y) is any function and the point $y^{(n)}$ is a rotation of the point $(x, y) = (x^{(0)}, y^{(0)})$ by an angle $\frac{n\pi}{3}$ counter-clockwise i.e.

$$\begin{aligned} f(x, y) = &\, rh(x, y) + sh(x^{'}, y^{'}) + th(x^{''}, y^{''}) \\ &+ uh(-x, -y) + vh(-x^{'}, -y^{'} + wh(-x^{''}, -y^{''}) \end{aligned} \tag{16}$$

where r, s , t , u , v and w are real numbers.

Dynamical systems with P6mm Symmetry (contd.)

From (15) and (16), we get t=s-r, u = -r, v = -s, and w = r-s. Therefore,

$$f(x,y) = r[h(x,y) - h(-x,-y)] + s[h(x^{'},y^{'}) - h(-x^{'},-y^{'})] \\ + (s-r)[h(x^{''},y^{''}) - h(-x^{''},-y^{''})]. \quad (17)$$

From (13), we have

$$g(x,y) = \frac{1}{\sqrt{3}} f(x,y) - \frac{2}{\sqrt{3}} f(x^{'},y^{'}). \quad (18)$$

Since the pattern also has a reflection in a line (x-axis), the function h(x,y) chosen should satisfy (6). From the periodic property of the pattern, we obtain from (3)

$$f(x,y) = f(x+T,y) = f(x,y+\alpha T) \\ g(x,y) = g(x+T,y) = g(x,y+\alpha T) \quad (19)$$

where $\alpha = \sqrt{3}$ or $\frac{1}{\sqrt{3}}$.

Dynamical systems with P6mm Symmetry (contd.)

Considering the possible choices of $h(x, y)$, and assuming that $h(x, y)$ is periodic along the x-axis with period 2π and the y-axis with period $2\sqrt{3}\pi$. Then, $h(x, y)$ may be expressed in Fourier series as

$$\begin{aligned} h(x,y) = \sum a_{mn} cos(mx) cos(\frac{ny}{\sqrt{3}}) + \sum b_{mn} cos(mx) sin(\frac{ny}{\sqrt{3}}) \\ + \sum c_{mn} sin(mx) cos(\frac{ny}{\sqrt{3}}) + \sum d_{mn} sin(mx) sin(\frac{ny}{\sqrt{3}}). \end{aligned} \quad (20)$$

From (6) and (17), the first, second and fourth sums on R.H.S. vanish. Therefore,

$$h(x,y) = \sum c_{mn} sin(mx) cos(\frac{ny}{\sqrt{3}}) \quad (21)$$

Dynamical systems with P6mm Symmetry (contd.)

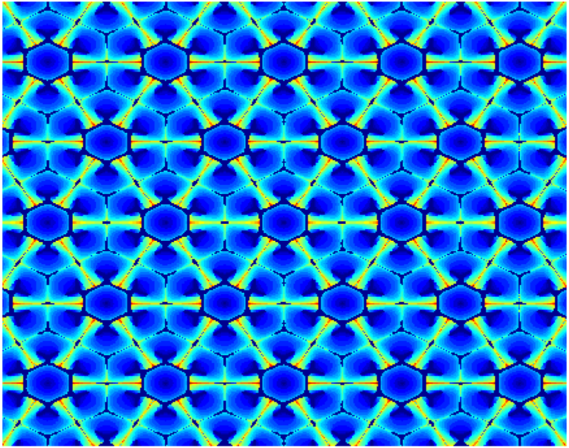
Assuming

$$h(x, y) = sin(x)cos(\frac{2y}{\sqrt{3}}) \tag{22}$$

we calculate f(x,y) and g(x,y) using (17,18) which are then substituted in (1).

Dynamical systems with P6mm Symmetry (contd.)

The corresponding pattern was programmed in Matlab and the result is shown in the figure below.



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Questions and Feedback..