Fourier Analysis Project: Hilbert Transform

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We are going to introduce the Hilbert transform in a couple of different ways. We are going to deal with the Hilbert transform in the sense of L^2 space, but before doing that, we can see how it works in the sense of distributions. In this report, labeling follows from that in the text. We start by considering the principal value of $1/x$.

Project 8.7.(a). (The Principal Value Distribution $1/x$) The linear functional $H_0 : \mathcal{S}(\mathbb{R}) \to \mathbb{C}$ is given by

$$H_0(\phi) = p.v. \int_{\mathbb{R}} \frac{\phi(y)}{y} dy := \lim_{\epsilon \to 0} \int_{|y|>\epsilon} \frac{\phi(y)}{y} dy.$$

Prove that the functional H_0 is continuous, so it is a tempered distribution. Also, find f and k such that H_0 is the k th derivative (in the sense of distributions) of f.

Proof. It suffices to show that $H_0(\phi_k) \to 0$ as $k \to \infty$, when $\phi_k \to 0$ in $\mathcal{S}(\mathbb{R})$ as $k \to \infty$. Observe that for each $\epsilon > 0$, $\int_{\epsilon<|y|<1} \frac{1}{y} dy = 0$. It follows that $\phi_k(0) \int_{\epsilon<|y|<1} \frac{1}{y} dy = 0$. We know that

$$\begin{aligned}
\left| \int_{|y|>\epsilon} \frac{\phi_k(y)}{y} dy \right| &= \left| \int_{|y|>\epsilon} \frac{\phi_k(y)}{y} dy - 0 \right| = \left| \int_{|y|>\epsilon} \frac{\phi_k(y)}{y} dy - \phi_k(0) \int_{\epsilon<|y|\le 1} \frac{1}{y} dy \right| \\
&\le \left| \int_{\epsilon<|y|\le 1} \frac{\phi_k(y) - \phi_k(0)}{y} dy \right| + \left| \int_{|y|>1} \frac{\phi_k(y)}{y} dy \right|.
\end{aligned}$$

Note that there exists c_y between y and 0 satisfying $\frac{\phi_k(y)-\phi_k(0)}{y} = \phi_k'(c_y)$, by the Mean Value Theorem. So, the first integral is

$$\left| \int_{\epsilon<|y|\le 1} \frac{\phi_k(y) - \phi_k(0)}{y} dy \right| = \left| \int_{\epsilon<|y|\le 1} \phi_k'(c_y) dy \right| \le \int_{\epsilon<|y|\le 1} \rho_{0,1}(\phi_k) dy \le 2 \int_0^1 \rho_{0,1}(\phi_k) dy = 2\rho_{0,1}(\phi_k).$$

Similarly, the second integral is

$$\left| \int_{|y|>1} \frac{\phi_k(y)}{y} dy \right| \le \int_{|y|>1} \frac{|\phi_k(y)|}{|y|} dy \le 2 \int_1^\infty \frac{\rho_{1,0}(\phi_k)}{y^2} dy = 2\rho_{1,0}(\phi_k) \left. -\frac{1}{y} \right|_1^\infty = 2\rho_{1,0}(\phi_k).$$

It follows that

$$\left| \int_{|y|>\epsilon} \frac{\phi_k(y)}{y} dy \right| \le 2(\rho_{1,0}(\phi_k) + \rho_{0,1}(\phi_k)).$$

If $\phi_k \to 0$ in $\mathcal{S}(\mathbb{R})$ as $k \to \infty$, then both $\rho_{0,1}(\phi_k)$ and $\rho_{1,0}(\phi_k)$ tend to 0 as $k \to \infty$. Thus, H_0 is continuous; it is a tempered distribution.

Now, we show that $H_0 = p.v.(1/x) = \frac{d}{dx}(\log|x|)$, that is, $f(x) = \log|x|$ and $k = 1$. Indeed, we know that

$$\begin{aligned}(\log|x|)'(\phi) &= -(\log|x|)(\phi') \\ &= -\int_{\mathbb{R}} \log|x|\phi'(x)dx \\ &= -\lim_{\epsilon\to 0}\int_{|x|>\epsilon} \log|x|\phi'(x)dx.\end{aligned}$$

Note that

$$\begin{aligned}\int_{\epsilon}^{\infty} \log|x|\phi'(x)dx &= [\log|x|\phi(x)]_{\epsilon}^{\infty} - \int_{\epsilon}^{\infty} \frac{\phi(x)}{x}dx \\ &= -\log(\epsilon)\phi(\epsilon) - \int_{\epsilon}^{\infty} \frac{\phi(x)}{x}dx.\end{aligned}$$

Similarly, we get $\int_{-\infty}^{-\epsilon} \log|x|\phi'(x)dx = \log(\epsilon)\phi(-\epsilon) - \int_{-\infty}^{-\epsilon} \frac{\phi(x)}{x}dx$. Since $\phi \in \mathcal{S}(\mathbb{R})$, we get $\phi(\epsilon) - \phi(-\epsilon) = 2\epsilon\phi'(c_\epsilon)$, by the Mean Value Theorem(c_ϵ is between $-\epsilon$ and ϵ). Using the L'Hospital Rule, we have that $\lim\limits_{\epsilon\to 0} \log\epsilon(\phi(\epsilon) - \phi(-\epsilon)) = 2\lim\limits_{\epsilon\to 0}(\epsilon\log\epsilon)\phi'(c_\epsilon) = 2\cdot 0\cdot\phi'(0) = 0$. Hence,

$$\begin{aligned}(\log|x|)'(\phi) &= -\lim_{\epsilon\to 0}\int_{|x|>\epsilon} \log|x|\phi'(x)dx \\ &= \lim_{\epsilon\to 0}\left[\log\epsilon(\phi(\epsilon) - \phi(-\epsilon)) + \int_{|x|>\epsilon} \frac{\phi(x)}{x}dx\right] \\ &= \lim_{\epsilon\to 0}\left[0 + \int_{|x|>\epsilon} \frac{\phi(x)}{x}dx\right] \\ &= H_0(\phi).\end{aligned}$$

Since $(\log|x|)'(\phi) = H_0(\phi)$ for every $\phi \in \mathcal{S}(\mathbb{R})$, we get $H_0 = p.v.(1/x) = \frac{d}{dx}(\log|x|)$, as desired. This completes the proof. ☐

So, we can define the principal value of $1/x$ in the sense of distributions.

Project 8.7.(b). For each $\epsilon > 0$, the function $x^{-1}\chi_{|x|>\epsilon}(x)$ defines a tempered distribution, which we call H_0^ϵ. Then, for each $\phi \in \mathcal{S}(\mathbb{R})$, we get

$$\begin{aligned}\lim_{\epsilon\to 0} H_0^\epsilon(\phi) &= \lim_{\epsilon\to 0}\int_{\mathbb{R}} x^{-1}\chi_{|x|>\epsilon}(x)\phi(x)dx \\ &= \lim_{\epsilon\to 0}\int_{|x|>\epsilon} \frac{\phi(x)}{x}dx \\ &= H_0(\phi).\end{aligned}$$

Now, we are ready to define the Hilbert transform in the sense of distributions.

Project 8.7.(c). For each $x \in \mathbb{R}$ define a new tempered distribution by appropriately translating and reflecting H_0, as follows.

Definition 8.57. (The Hilbert Transform as a Distribution). Given $x \in \mathbb{R}$, the Hilbert transform $H_x(\phi)$ of ϕ at x, also written $H\phi(x)$, is a tempered distribution acting on $\phi \in \mathcal{S}(\mathbb{R})$ and defined by

$$H_x(\phi) = H\phi(x) := (\tau_{-x}H_0)^{\sim}(\phi)/\pi.$$

Verify that $H\phi(x) = \frac{1}{\pi}\lim_{\epsilon\to 0}\int_{|x-t|>\epsilon} \frac{\phi(t)}{x-t}dt$.

Proof. Note that $\tau_x\widetilde{\phi}(y) = \tau_x(\widetilde{\phi}(y)) = \widetilde{\phi}(y-x) = \phi(x-y)$. Using the time-frequency dictionary, we have that

$$\begin{aligned}
H\phi(x) &= \frac{1}{\pi}(\tau_{-x}H_0)^{\sim}(\phi) = \frac{1}{\pi}(\tau_{-x}H_0)(\widetilde{\phi}) = \frac{1}{\pi}H_0(\tau_x(\widetilde{\phi})) \\
&= \frac{1}{\pi}\lim_{\epsilon\to 0}\int_{|y|>\epsilon}\frac{\tau_x\widetilde{\phi}(y)}{y}dy \\
&= \frac{1}{\pi}\lim_{\epsilon\to 0}\int_{|y|>\epsilon}\frac{\phi(x-y)}{y}dy \\
&= \frac{1}{\pi}\lim_{\epsilon\to 0}\int_{|x-t|>\epsilon}\frac{\phi(t)}{x-t}dt.
\end{aligned}$$

Here, the last equality holds by substituting $y = x - t$ and break the integral into two pieces so that "$-$" signs are canceled with each other. □

In fact, we can replace $\mathcal{S}(\mathbb{R})$ by $L^2(\mathbb{R})$ in the definition of the Hilbert transform.

The first definition is,

Definition 12.12. The Hilbert transform H is defined by

$$Hf(x) := \lim_{\epsilon\to 0}\frac{1}{\pi}\int_{|x-y|>\epsilon}\frac{f(y)}{x-y}\,dy\ for\ f \in L^2(\mathbb{R}).$$

Here, this definition is well-defined as follows:

Set $k_{\epsilon,R}(y) := (1/\pi y)\chi_{\{y\in\mathbb{R}:\epsilon<|y|<R\}}(y)$. Since $k_{\epsilon,R}$ is odd, we get $(k_{\epsilon,R})^\wedge(0) = 0 = i\,sgn(0)$. For $\xi \neq 0$, note that

$$\begin{aligned}
(k_{\epsilon,R})^\wedge(\xi) &= \int_{\epsilon<|y|<R}\frac{1}{\pi y}e^{-2\pi i y\xi}\,dy \\
&= -i\,sgn(\xi)\frac{2}{\pi}\int_{2\pi\epsilon|\xi|}^{2\pi R|\xi|}\frac{\sin t}{t}\,dt \to -i\,sgn(\xi),
\end{aligned}$$

as $\epsilon \to 0,\ R \to \infty$.

Set $H_{\epsilon,R}f := k_{\epsilon,R} * f = \frac{1}{\pi}\int_{\epsilon<|x-y|<R}\frac{f(y)}{x-y}\,dy$. Then, $H_{\epsilon,R}f \to Hf$, as $\epsilon \to 0,\ R \to \infty$. Also, we get $(H_{\epsilon,R}f)^\wedge(\xi) = (k_{\epsilon,R})^\wedge(\xi)\widehat{f}(\xi) \to -i\,sgn(\xi)\widehat{f}$, as $\epsilon \to 0,\ R \to \infty$. By the continuity of the Fourier transform, it follows that $(Hf)^\wedge = -i\,sgn(\xi)\widehat{f}$.

Since the Fourier transform in $L^2(\mathbb{R})$ is well-defined, so is $(Hf)^\wedge$. So, by the Inversion formula, Hf is well-defined. Here, note that the well-definedness of Hf is equivalent to that of $(Hf)^\wedge$. This induces another definition of the Hilbert transform:

Definition 12.1. The Hilbert transform H is defined on the Fourier side by the formula

$$(Hf)^\wedge(\xi) := -i\,sgn(\xi)\widehat{f}(\xi),$$

for f in $L^2(\mathbb{R})$.

Historically, this is the definition of the Hilbert transform to solve the following problem:

Theorem (The Riemann-Hilbert Problem). Given f defined on $\mathbb{R}$, find holomorphic functions F^+ and F^- defined on the upper and lower half-planes, respectively, such that $f = F^+ - F^-$.

Here, the Hilbert transform is given by $Hf = \frac{1}{i}(F^+ - F^-)$.

Let me list properties of the Hilbert transform here:

- $\widehat{f}(\xi) + i\widehat{Hf}(\xi)$ is equal to $2\widehat{f}(\xi)$ for $\xi > 0$ and zero for $\xi < 0$.
- The Hilbert transform is an isometry on $L^2(\mathbb{R})$.
- Commutativity(translation): $\tau_h H = H\tau_h$, where $\tau_h f(x) := f(x-h)$.
- Commutativity(dilation): $\delta_a H = H\delta_a$, where $\delta_a f(x) := f(ax)$, for $a > 0$.
- Anti commutativity(reflection): $\widetilde{Hf} = -H\widetilde{f}$, where $\widetilde{f}(x) := f(-x)$.

The first property follows from Definition 12.1. The second property is proved by applying Plancherel's Identity twice. The last three (Anti-)Commutative properties can be proved when we look at those identities on the Fourier side by recalling that the Fourier transform is a bijection on $L^2(\mathbb{R})$. Notice that the second property tells us that the Hilbert transform is bounded. The following proposition says that the last 4 properties above "define" the Hilbert transform.

Proposition (Commutativity). Let T be a bounded operator on $L^2(\mathbb{R})$ that commutes with translations and dilations. If T anticommutes with reflections, then T is a constant multiple of the Hilbert transform: $T = cH$ for some $c \in \mathbb{R}$.

In fact, the above fact is just a part of Exercise 12.24. in the text. This position tells us that if we can find a bounded operator on $L^2(\mathbb{R})$ satisfying the two commutative properties and the one anti-commutative property, then the operator is a multiple of the Hilbert transform. In the sense of the Haar basis, the average of the dilated and translated dyadic shift operators does satisfies all those properties. Indeed, if we pick this operator as our T in the proposition, then we can say $c = -\frac{8}{\pi}$, specifically. This process gives us the third definition of the Hilbert transform in the L^2 sense.

We have seen the definitions of the Hilbert transform so far, one as a distribution, three as a operator on $L^2(\mathbb{R})$. Now, here is a different approach: considering a kernel in the complex plane.

Exercise 12.55. (Poisson and conjugate Poisson Kernels) Show that the Poisson kernel $P_y(x)$ and the conjugate Poisson kernel $Q_y(x)$ are given by $P_y(x) = \frac{y}{\pi(x^2+y^2)}$, $Q_y(x) = k\frac{x}{\pi(x^2+y^2)}$. Calculate the Fourier transform of $Q_y(x)$ for each $y > 0$, and show that $\widehat{Q_y(\xi)} = -isgn(\xi)exp(-2\pi|y\xi|)$. Show that as $y \to 0$, $Q_y(x)$ approaches the principal value distribution $p.v.\frac{1}{\pi x}$.

Proof. Let $f \in L^2(\mathbb{R})$ and $F(z)$ be twice the analytic extension of f to the upper half plane $\mathbb{R}^2_+ = \{z = x+iy : y > 0\}$. By using Cauchy integral formula, $F(z)$ can be written as:

$$F(z) = \frac{1}{\pi i}\int_{\mathbb{R}} \frac{f(t)}{t-z}dt, \qquad z \in \mathbb{R}^2_+. \tag{1}$$

Putting $z = x+iy, y > 0$ in (1), we obtain the following expression:

$$\begin{aligned}
F(z) &= \frac{1}{\pi i}\int_{\mathbb{R}} \frac{f(t)}{(t-x)-iy}dt \\
&= \frac{1}{\pi i}\int_{R} \frac{((t-x)+iy}{(t-x)^2+y^2}f(t)dt \\
&= \frac{1}{\pi i}\int_{\mathbb{R}} \frac{(t-x)}{(t-x)^2+y^2}f(t)dt + \frac{1}{\pi}\int_{\mathbb{R}} \frac{y}{(t-x)^2+y^2}f(t)dt \\
&= \frac{i}{\pi}\int_{\mathbb{R}} \frac{(x-t)}{(x-t)^2+y^2}f(t)dt + \frac{1}{\pi}\int_{\mathbb{R}} \frac{y}{(x-t)^2+y^2}f(t)dt \\
&= iQ_y(x) * f + P_y(x) * f. \\
&= \frac{i}{\pi}\int_{\mathbb{R}} \frac{(x-t)}{(x-t)^2+y^2}f(t)dt + \frac{1}{\pi}\int_{\mathbb{R}} \frac{y}{(x-t)^2+y^2}f(t)dt \\
&= iQ_y(x) * f + P_y(x) * f.
\end{aligned}$$

Thus we have $P_y(x) = \frac{y}{\pi(x^2+y^2)}$ and $Q_y(x) = k\frac{x}{\pi(x^2+y^2)}$.
Now we find the Fourier transform of $Q_y(x)$. We have $Q_y(x) = \frac{x}{\pi(x^2+y^2)}$. Since $Q_y(x) \notin L^1(\mathbb{R})$ and does not have moderate decay, we define its Fourier transform by following:

$$\begin{aligned}
\widehat{Q_y}(\xi) &= p.v \int_{\mathbb{R}} Q_y(x) e^{-2\pi i x\xi} dx. \\
&= p.v \frac{1}{\pi} \int_{\mathbb{R}} \frac{xe^{-2\pi i x\xi}}{x^2+y^2} dx \\
&= \lim_{R\to\infty} \frac{1}{\pi} \int_{-R}^{R} \frac{xe^{-2\pi i x\xi}}{x^2+y^2} dx. \qquad (2)
\end{aligned}$$

We use the contour integral to evaluate (2).
Let

$$g(z) = \frac{ze^{-2\pi i z\xi}}{\pi(z^2+y^2)}, \xi \in \mathbb{R}, y > 0$$

and we see $z = \pm iy$ are the simple poles of $g(z)$.

For $\xi < 0$ choose $\Gamma_R^+ = C_R^+ \cup [-R, R]$ where C_R^+ is the upper half semicircle of radius R and centred at origin which contains the point of singularity $z = iy$ of $g(z)$.

$$Res(g(z), iy) = \lim_{z\to iy} \frac{(z+iy)ze^{-2\pi i x\xi}}{\pi(z^2+y^2)} = \frac{e^{2\pi y\xi}}{2\pi} \quad for \quad \xi < 0$$

For $\xi > 0$ choose $\Gamma_R^- = C_R^- \cup [-R, R]$ where C_R^+ is the lower half semicircle of radius R and centred at origin which contains the point of singularity $z = -iy$ of $g(z)$.

$$Res(g(z), -iy) = \lim_{z\to -iy} \frac{(z-iy)ze^{-2\pi i x\xi}}{\pi(z^2+y^2)} = \frac{e^{-2\pi y\xi}}{2\pi} \quad for \quad \xi > 0$$

$$\begin{aligned}
\widehat{Q_y}(\xi) &= \lim_{R\to\infty} \frac{1}{\pi} \int_{-R}^{R} \frac{xe^{-2\pi x\xi}}{x^2+y^2} dx = \lim_{R\to\infty} \oint_{\Gamma_R^+} g(z)dz \\
&= 2\pi i Res(g(z), iy) \\
\widehat{Q_y(\xi)} &= ie^{2\pi y\xi}. \qquad for \quad \xi < 0.
\end{aligned}$$

Similarly,

$$\widehat{Q_y(\xi)} = -ie^{-2\pi y\xi} \qquad for \quad \xi > 0.$$

Therefore we conclude that

$$\widehat{Q_y(\xi)} = -isgn(\xi)e^{-2\pi y|\xi|} \qquad (3)$$

where $y > 0$ and $sgn(\xi) = 1$ if $\xi > 0$, -1 if $\xi < 0$ and 0 if $\xi = 0$. □

Remark 1. From (3) we see that $\lim_{y\to 0} \widehat{Q_y(\xi)} = -isgn(\xi)$.

Remark 2. Note that

$$\begin{aligned}
\lim_{y\to 0} Q_y * f(x) &= \lim_{y\to 0} \frac{1}{\pi} \int_{\mathbb{R}} \frac{(x-t)^2}{(x-t)^2+y^2} f(t)dt \\
&= p.v. \frac{1}{\pi} \int_{\mathbb{R}} \frac{1}{x-t} f(t)dt = Hf(x).
\end{aligned}$$

By Remark 1 and continuity of Fourier transform in $L^2(\mathbb{R})$ we conclude that

$$(Hf)^\wedge(\xi) = -isgn(\xi)\hat{f}(\xi).$$

Remark 3. In project 7.8 we proved that Poisson kernel is an approximation of identity. Therefore we have $\lim_{y\to 0} P_y * f(x) = f(x)$ in L^2 .

Definition 12.7.2. The Fourier multiplier T_m of a periodic function $f \in L^2(\mathbb{T})$ on the Fourier side is given by

$$(T_m f)^\wedge(n) = m(n)\hat{f}(n).$$

where the sequence $\{m(n)\}_{n\in\mathbb{Z}}$ is called the symbol of T_m. For $H_p, m(n) = -isgn(n)$.

So, the Fourier multiplier of the Hilbert transform can be viewed as the limit of the conjugate Poisson kernel Q_y.

Now, we move on to the next section: Connection to Fourier series. We need the definition of the partial Fourier sum for $L^2(\mathbb{T})$

Definition . (Partial Fourier sum) For a nice function $f \in L^2(\mathbb{T})$, the N^{th} partial Fourier sum is given by

$$S_N f(\theta) = \sum_{|n|\leq N} \hat{f}(n) e^{2\pi i n\theta}.$$

We can prove that $S_N f$ is a Fourier multiplier with symbol $m_N(n) = 1$ if $|n| \leq N$ and $m_N(n) = 0$ if $|n| > N$.

Exercise 12.60. (Partial Fourier Sums as Modulations of the Periodic Hilbert transform) We prove 2 and 3 of the the followings(1 follows from simple calculations. In the talk, we used "a picture" to show 1):

1. $$m_N(n) = \begin{cases} (sgn(n+N) - sgn(n-N))/2, & \text{if}|n| \neq N \\ sgn(n+N) - sgn(n-N), & \text{if}|n| = N \end{cases}.$$

2. Let $f \in L^p(\mathbb{T})$ and M_N denote the modulation operator $M_N f(\theta) := f(\theta)e^{2\pi i\theta N}$. Show that

$$i(M_N H_p M_{-N})^\wedge(n) = sgn(n+N)\hat{f}(n).$$

3. Let M_N denote the modulation operator ,$M_N f(\theta) := f(\theta)e^{2\pi i\theta N}$.Then

$$S_N f(\theta) = \frac{i}{2}(M_{-N}H_p M_N f(\theta) - M_N H_p M_{-N} f(\theta)) + \frac{1}{2}(\hat{f}(N)e^{2\pi i N\theta} + \hat{f}(-N)e^{-2\pi i N\theta}).$$

Proof. The second statement follows from the time-frequency dictionary:

$$\begin{aligned}(M_N H_p M_{-N} f(\theta))^\wedge(n) &= \tau_N (H_p M_{-N} f(\theta))^\wedge(n) \\ &= (H_p M_{-N} f(\theta))^\wedge(n-N) \\ &= -isgn(n-N)\widehat{M_{-N} f(\theta)}(n-N) \\ &= -isgn(n-N)\tau_{-N}\hat{f}(n-N) \\ &= -isgn(n-N)\hat{f}(n).\end{aligned}$$

Now, the N^{th} Fourier partial sum of a function f is

$$\begin{aligned}
S_N f(\theta) &= \sum_{|n|\leq N} \hat{f}(n)e^{2\pi in\theta} \\
&= \sum_{n\in\mathbb{Z}} \hat{f}(n)e^{2\pi in\theta}\chi_{[-N,N]} \\
&= \sum_{n\in\mathbb{Z}} m_N(n)\hat{f}(n)e^{2\pi in\theta} \\
&= \sum_{|n|\neq N} m_N(n)\hat{f}(n)e^{2\pi in\theta} + \sum_{|n|=N} m_N(n)\hat{f}(n)e^{2\pi in\theta} \\
&= \sum_{|n|\neq N} m_N(n)\hat{f}(n)e^{2\pi in\theta} + \frac{1}{2}\sum_{|n|=N} m_N(n)\hat{f}(n)e^{2\pi in\theta} + \frac{1}{2}\sum_{|n|=N} m_N(n)\hat{f}(n)e^{2\pi in\theta}.
\end{aligned}$$

By using 1 of Exercise(12.60) above we get

$$\begin{aligned}
S_N f(\theta) = &\sum_{|n|\neq N} (\frac{sgn(n+N) - sgn(n-N)}{2}\hat{f}(n)e^{2\pi in\theta} + \sum_{|n|=N} (\frac{sgn(n+N) - sgn(n-N)}{2}\hat{f}(n)e^{2\pi in\theta} \\
&+ \frac{1}{2}\sum_{|n|=N} m_N(n)\hat{f}(n)e^{2\pi in\theta} \\
&= \sum_{n\in\mathbb{Z}} (\frac{sgn(n+N) - sgn(n-N)}{2}\hat{f}(n)e^{2\pi in\theta} + \frac{1}{2}\sum_{|n|=N} m_N(n)\hat{f}(n)e^{2\pi in\theta}.
\end{aligned}$$

By using 2 of Exercise(12.60) we get

$$\begin{aligned}
S_N f(\theta) = &\frac{i}{2}\sum_{n\in\mathbb{Z}} (M_{-N}H_pM_Nf(\theta))^\wedge(n)\hat{f}(n)e^{2\pi in\theta} - \frac{i}{2}\sum_{n\in\mathbb{Z}} (M_NH_pM_{-N}f(\theta))^\wedge(n)\hat{f}(n)e^{2\pi in\theta} \\
&+ \frac{1}{2}\{m_N(N)\hat{f}(N)e^{2\pi iN\theta} + m_N(-N)\hat{f}(-N)e^{-2\pi iN\theta}\}.
\end{aligned}$$

Using 1 Exercise(12.60) again we conclude that

$$S_N f(\theta) = \frac{i}{2}(M_{-N}H_pM_Nf(\theta) - M_NH_pM_{-N}f(\theta)) + \frac{1}{2}(\hat{f}(N)e^{2\pi iN\theta} + \hat{f}(-N)e^{-2\pi iN\theta}).$$

This completes the proof. □

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