ON THE RESOLVENT OF THE DYADIC PARAPRODUCT, AND A NONLINEAR OPERATION ON RH_p WEIGHTS

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ABSTRACT. The existence of a bounded inverse of $(I - \Pi_b)$ on L^p (Π_b is the dyadic paraproduct) does not imply the same for $(I - \lambda\Pi_b)$, $-1 \le \lambda < 1$ (we present a counterexample); but it guarantees the existence of $1 < p_o$ such that there exist a bounded inverse in L^{p_o} for every $-1 \le \lambda \le 1$. This is equivalent to showing that the RH_p^d class of weights is not preserved under certain nonlinear operation involving λ, but if $\omega \in RH_p^d$ then there exists $1 < p_o$ such that the transformed weight $\omega_\lambda \in RH_{p_o}^d$ for all $-1 \le \lambda \le 1$.

1. INTRODUCTION

The necessary and sufficient conditions for the existence of a solution $f \in L^p(\mathbf{R})$ of the equation:

$$(I - \Pi_b)f = g, \qquad \|f\|_p \le C\|g\|_p; \tag{1}$$

are known (see [8]). Here I is the identity operator, Π_b is the dyadic paraproduct (see (3) for the precise definition), associated to the function b of bounded mean oscillation (BMO), and g is a function in $L^p(\mathbf{R})$.

We are interested in studying this equation under a simple perturbation:

$$(I - \lambda\Pi_b)f = g, \qquad -1 \le \lambda \le 1. \tag{2}$$

It is known, by spectral theory, that as soon as (1) is solvable, then so is (2) in a neighborhood of $\lambda = 1$; clearly the same is true near $\lambda = 0$. What about $0 < \lambda < 1$? Does invertibility of $(I - \Pi_b)$ in L^p guarantees invertibility of $(I - \lambda\Pi_b)$ for $0 < \lambda < 1$?

The answer to this question is no. Counterexamples were constructed for $-1 < \lambda < 0$ (see [8]) it was not clear then what the answer was for $0 < \lambda < 1$.

The questions we are asking can be rephrased in terms of preserving reverse Hölder p (RH_p) weights (see [6]) under certain nonlinear operation involving λ (see §3 for definitions and precise statements).

Theorem 1.0.1. There exist a doubling weight $\omega \in RH_p^d$, $1 < p < \infty$, and $0 < \lambda < 1$, such that $\omega_\lambda \notin RH_p^d$ (ω_λ will be defined later).

The weight ω_λ is given by an infinite product where λ appears in each factor (see (5)). Inserting $\lambda = 1$ we get back the weight ω we started with. Unfolding the product, you get $\omega_\lambda = 1 + \lambda b + \ldots$ where $b \in BMO$. At first sight we are

1991 *Mathematics Subject Classification.* Primary 42c; Secondary 47b.

Key words and phrases. RH_p weights, dyadic paraproduct.

The author was supported by NSF grant #DMS 9304580.

tempted to say that ω_λ plays a role much like the more traditional $\omega^\lambda = e^{\lambda b}$ does. Theorem 1.0.1 makes clear the substantial differences between them, since if $\omega \in RH_p^d$ then $\omega^\lambda \in RH_p^d$ for all $0 \le \lambda \le 1$ (this is just a trivial application of Hölder's inequality!).

To the weight ω given by Theorem 1.0.1 we can associate a function $b \in BMO$, such that to ω_λ corresponds the function $\lambda b \in BMO$, and $(I - \Pi_b)$ is invertible in L^p, but $(I - \lambda\Pi_b) = (I - \Pi_{\lambda b})$ is not.

Nevertheless we will show that:

Theorem 1.0.2. There exists $p_0 > 1$ such that if $(I - \Pi_b)$ is invertible in L^p, then $(I - \lambda\Pi_b)$ is invertible in L^{p_0} for all $-1 \le \lambda \le 1$.

Notation and basic definitions are in §2. In §3 we recall a dyadic characterization of weights, the correspondence we mentioned between weights and functions in BMO becomes clear. In §4 we recall the necessary and sufficient conditions for inverting $(I - \lambda\Pi_b)$ in L^p, and we prove Theorem 1.0.2. In the last section we construct the counterexample.

Throughout this paper C will denote a constant that might change from line to line.

We would like to express here our inmense gratitude to Mischa Cotlar for a life dedicated to people and mathematics, thanks Mischa for being.

2. Preliminaries

2.1. **Dyadic intervals and Haar basis.** We will work on $\mathbf{R}$ but everything holds in $\mathbf{R}^n$ (see [8]).

Let us denote by $\mathcal{D}$ the family of all dyadic intervals in $\mathbf{R}$, i.e. intervals of the form $(j2^{-k}, (j+1)2^{-k}]$, $j, k \in \mathbf{Z}$. $\mathcal{D}_k$ denotes the k^{th} generation of $\mathcal{D}$, consisting of those dyadic intervals of length 2^{-k}. Given any interval J, $\mathcal{D}(J)$ denotes the family of dyadic subintervals of J; $\mathcal{D}_n(J) = \{I \in \mathcal{D}(J) : |I| = 2^{-n}|J|\}$, $|I|$ denotes the length of the interval I. Given an interval J we will denote the right and left halves respectively by J_r and J_l; they are the elements of $\mathcal{D}_1(J)$.

The *Haar function* associated to an interval I is given by:

$$h_I = \frac{1}{|I|^{1/2}}(\chi_{I_r}(x) - \chi_{I_l}(x)),$$

here χ_I denotes the characteristic function of the interval I.

The Haar functions indexed on the dyadics, $\{h_I\}_{I\in\mathcal{D}}$, form a basis of $L^2(\mathbf{R})$.

2.2. **Expectation and Difference operators.** We define the expectation and difference operators for locally integrable functions by:

$$\begin{aligned} E_k f(x) &= \frac{1}{|I|}\int_I f(t)dt = m_I f, \qquad x \in I \in \mathcal{D}, \\ \Delta_k f(x) &= E_{k+1}f(x) - E_k f(x). \end{aligned}$$

As operators defined on L^p, it is clear that $E_k = \sum_{j<k} \Delta_j$, and $\sum_j \Delta_j$ =identity operator. It is not hard to check that:

$$\Delta_k f(x) = \sum_{I\in\mathcal{D}_k} \langle f, h_I\rangle h_I(x),$$

here $\langle .,. \rangle$ denotes the standard inner product in L^2. This proves that the Haar system is complete.

2.3. **Dyadic paraproducts and BMO.** A locally integrable function b is in the space of *bounded mean oscillation*, BMO, if

$$\frac{1}{|J|}\int_J |b(x) - m_J b|^2 \leq C|J|,$$

for every interval J, recall that $m_J b = \frac{1}{|J|}\int_J b$. This is equivalent to the *Carleson condition*

$$b \in BMO \iff \sum_{I\in\mathcal{D}(J)} b_I^2 \leq C|J| \quad \forall J,$$

where here $b_I = \langle b, h_I\rangle$. (It is just an application of Plancherel's Theorem for orthonormal basis.)

Define formally the *dyadic paraproduct* associated to a function $b \in BMO$ by:

$$\Pi_b f = \sum_k E_k f \Delta_k b. \tag{3}$$

The dyadic paraproduct is a bilinear operator known to be bounded in L^p, more precisely,

$$\|\Pi_b f\|_p \leq C\|b\|_{BMO}\|f\|_p.$$

See [7], or [2], for more about BMO, paraproducts and related subjects.

2.4. **Weights.** A *doubling weight* ω is a positive locally integrable function such that $\omega(2I) \leq C\omega(I)$ for all intervals I. (We are using the notations $\omega(I) = \int_I \omega$, and $2I$ is an interval concentric to I and with double length.)

A weight ω is in A_∞ if given $\epsilon > 0$, there exists $\delta > 0$ such that for any interval I, $E \subset I$, such that $|E| < \delta|I|$, then $\omega(E) \leq \epsilon\omega(I)$. (Eg: $\omega(x) = |x|^\alpha$ for $-1 < \alpha$.)

Every A_∞ weight is doubling, but the converse is not true (see [4],[9]). There are equivalent definitions of A_∞, for example:

A weight ω is in A_∞ if for every interval I

$$\frac{1}{|I|}\int_I \omega \leq C\exp\left(\frac{1}{|I|}\int_I \ln\omega\right).$$

The smallest of such C's is called the A_∞ constant of the weight ω.

See [6] for the general theory of weights.

A weight ω is in RH_p (reverse Hölder p) if for every interval I

$$\left(\frac{1}{|I|}\int_I \omega^p\right)^{1/p} \leq C\frac{1}{|I|}\int_I \omega.$$

The smallest of such C's is called the RH_p constant of the weight ω.

(Eg: $\omega(x) = |x|^\alpha$ for $\alpha > -1/p$.)

The main properties of these classes of weights are the following:

(a) if $\omega \in RH_p$ then $\omega \in RH_{p+\epsilon}$ for some $\epsilon > 0$;
(b) if $p < q$ then $RH_q \subset RH_p$;
(c) $A_\infty = \cup_{p>1} RH_p$.

Property (b) is a trivial consequence of Hölder's inequality. Property (a) is a classical result of Gehring (see [5], [6]). And property (c) can be found in [6], Thm. 2.11.

Remark 1: The A_∞ constant of a weight ω forces a lower bound p_o, $p_o > 1$, on the range of p's such that ω is not in RH_p. This can be seen in the proof of property (c) (see [5], [6]).

There is a classical correspondence between A_∞ weights and functions in BMO. Namely, if $\omega \in A_\infty$ then $b = \log \omega \in BMO$. Conversely if, $b \in BMO$ and has sufficiently small BMO norm, then $e^b \in A_\infty$. We will consider a different correspondence in the next section, first introduced in [3].

3. Dyadic characterization of weights

In this section we will restrict our attention to functions defined on the unit interval $I_o = [0,1]$.

Let ω be a weight defined on I_o, such that $E_o\omega = m_{I_o}\omega = 1$. By the Lebesgue differentiation theorem, $\lim_{k\to\infty} E_k\omega(x) = \omega(x)$ for a.e. x; we can then write the telescoping product:

$$\omega(x) = \prod_{k=0}^{\infty} \frac{E_{k+1}\omega(x)}{E_k\omega(x)} = \prod_{k=0}^{\infty}\left(1 + \frac{\Delta_k\omega(x)}{E_k\omega(x)}\right).$$

Let us define the function b, at least formally, by

$$b = \sum_{k=0}^{\infty} \Delta_k b, \quad \Delta_k b = \frac{\Delta_k \omega}{E_k\omega}. \tag{4}$$

Still at the formal level, given a locally integrable function b, we can write:

$$\omega = \prod_{k=0}^{\infty} (1 + \Delta_k b). \tag{5}$$

The partial products are always defined and, in case of convergence, they correspond to the expectation of ω at level k, i.e.:

$$E_k\omega = \prod_{j=0}^{k-1} (1 + \Delta_j b). \tag{6}$$

Definition 3.0.1. A locally integrable function b is of RH_p^d-type if

$$\sum_{I\in\mathcal{D}(J)} m_I^p\omega\, b_I^2 \le C m_J^p\omega\,|J|, \ \ \forall J \in \mathcal{D}(I_o); \tag{7}$$

here $m_I\omega = \prod_{I\subset I'\in\mathcal{D}(I_o)}(1 + b_{I'}h_{I'}(x_I))$, where $x_I \in I$.

Under certain conditions, the formal equations (4) and (5) make sense. Properties of the weight ω can be read off properties on the corresponding function b, and viceversa. We have the following *dictionary*:

(i) if $|\Delta_k b| < 1$ then the partial products in (6) converge weakly to a positive measure.
(ii) $|\Delta_k b| < 1 - \epsilon$ for all $k \ge 0$ if and only if ω is a dyadic doubling weight.
(iii) $b \in BMO^d$ if and only if $\omega \in A_\infty^d$. The A_∞ constant of ω depends only on the BMO norm of b.
(iv) b is of RH_p^d-type if and only if $\omega \in RH_p^d$.

Remark 2: In this setting all conditions are *dyadic*, i.e. they hold on dyadic intervals, the superscript d indicates that (e.g. BMO^d, A_∞^d, etc).

The results (i)-(iii) appeared first in [3]. This characterization of the dyadic RH_p classes is due to S. Buckley (see [1]).

Remark 3: For the dyadic theory to resemble the classical theory (in particular if we want properties (a)-(c) to hold), we have to assume that the weight ω is dyadic doubling (i.e. $\omega(\tilde{I}) \leq C\omega(I)$, where $\tilde{I}$ is the dyadic parent of I); or equivalently that the corresponding b satisfies $|\Delta_k b| < 1-\epsilon$ (by (ii)).

4. Inverting $(I - \lambda\Pi_b)$

The following results are known (see [8]). We will state the theorems for functions defined on the unit interval $I_o = [0,1]$, in that case we must assume that the functions have mean value zero on I_o. The results are true for functions in $L^p(\mathbf{R})$. Let $L^p_o(I_o) = \{f \in L^p(I_o) : \int_{I_o} f = E_0 f = 0\}$.

When writing spaces of functions or classes of weights we will sometimes "forget" the domain of definition I_o, (e.g. L^p_o, RH^d_p, where it should read $L^p_o(I_o)$, $RH^d_p(I_o)$.)

Theorem 4.0.1. Given a locally integrable function b such that $|\Delta_k b| < 1-\epsilon$ for all $k \geq 0$, the operator $(I - \Pi_b)$ has a bounded inverse in $L^p_o(I_o)$ if and only if b is of RH^d_p-type. Moreover, we have an explicit formula for the inverse operator

$$(I - \Pi_b)^{-1} g(x) = \sum_{k=0}^{\infty} \Delta_k g(x) \prod_{j>k} (1 + \Delta_j b(x)).$$

Let $\omega = \prod_{j=0}^{\infty}(1+\Delta_j b)$. We can write, by (6),

$$\prod_{j>k}(1+\Delta_j b(x)) = \frac{\omega(x)}{E_k\omega(x)(1+\Delta_k b(x))}.$$

Given a doubling weight ω, define formally the operator

$$P_\omega g(x) = \sum_{k=0}^{\infty} \frac{\omega(x)\,\Delta_k g(x)}{E_k\omega(x)\,(1+\Delta_k b(x))},$$

(recall that $\Delta_k b = \Delta_k\omega / E_k\omega$).

Theorem 4.0.2. P_ω is a well defined and bounded operator in $L^p(I_o)$ if and only if $\omega \in RH^d_p(I_o)$.

The operator P_ω is an example of the *multiplier operator*:

$$Tf(x) = \sum_{k=0}^{\infty} \omega_k(x) \Delta_k f(x).$$

Clearly if the multipliers are constant functions, $\omega_k = a_k$, then T is bounded in L^p if and only if the sequence of a_k's is bounded. The general conditions on the sequence of multipliers that will guarantee boundedness of T are not known. The necessary and sufficient conditions are known for a few particular cases (see [8]).

Proof. [Sketch of the proof of Theorem 4.0.1] Suppose that f is a solution of the equation $f = g + \Pi_b f$. It is not hard to check that $\Delta_k f = \Delta_k g + E_k f \Delta_k b$, using the properties of the expectation and difference operators, and the definition of the dyadic paraproduct. Recall that $\Delta_k = E_{k+1} - E_k$, we obtain then the *recurrence equation*

$$E_{k+1} f = \Delta_k g + (1 + \Delta_k b) E_k f.$$

Solving the recurrence, and passing to the limit in L^p (using Theorem 4.0.2) we get that $f = P_\omega g$. □

The paraproduct is a bilinear operation, in particular $\lambda\Pi_b = \Pi_{\lambda b}$. Therefore, by the previous theorems, questions about the invertibility of $(I - \lambda\Pi_b)$ are reduced to questions about the weight

$$\omega_\lambda = \prod_{k=0}^{\infty}(1 + \lambda\Delta_k b), \tag{8}$$

corresponding to the function λb.

Given a doubling weight $\omega \in RH_p^d$, $b \in BMO$ corresponding to ω as described in the previous section, the operator $(I - \Pi_b)$ is invertible in L^p. By spectral theory (the resolvent is an open set of $\mathbf{C}$), we know that $(I - \lambda\Pi_b)$ will be invertible in neighborhoods of $\lambda = 1$ and $\lambda = 0$. Is this last statement true for $-1 < \lambda < 1$?

This question can be translated into a question about weights.

Does multiplication by $-1 < \lambda < 1$ on the b side preserves RH_p^d weights? Graphically:

$$\begin{array}{ccl} b & \longleftrightarrow & \omega \in RH_p^d \\ \lambda b & \longleftrightarrow & \omega_\lambda \in RH_p^d \quad ? \end{array}$$

The answer is negative. For $-1 < \lambda < 0$ counterexamples were constructed in [8]. We will present a counterexample for $0 < \lambda < 1$ in the last section.

If we replace RH_p^d by doubling A_∞^d, then the statement is true, more precisely:

Lemma 4.0.3. Given a doubling weight $\omega \in A_\infty^d$ then ω_λ is a doubling A_∞^d weight for every $-1 \leq \lambda \leq 1$. Moreover the A_∞ constants are uniformly bounded.

Proof. ω is a doubling A_∞^d weight $\iff b \in BMO$ and $|\Delta_k b| < 1 - \epsilon$ (properties (ii) and (iii)) $\Rightarrow \lambda b \in BMO$, and, since $-1 \leq \lambda \leq 1$, certainly $|\Delta_k \lambda b| = |\lambda \Delta_k b| < 1 - \epsilon$, and $\|\lambda b\|_{BMO} \leq \|b\|_{BMO} \iff \omega_\lambda$ is a doubling A_∞^d weight with A_∞ constant depending only on $\|b\|_{BMO}$. □

Nevertheless it is true that:

Theorem 4.0.4. Given a doubling weight $\omega \in RH_p^d$, then there exists $p_o > 1$ such that $\omega_\lambda \in RH_{p_o}^d$ for all $-1 \leq \lambda \leq 1$.

This implies that:

Theorem 4.0.5. Let b be a locally integrable function such that $|\Delta_k b| < 1 - \epsilon$ for all k. If $(I - \Pi_b)$ has a bounded inverse in L^p then there exists $p_o > 1$ such that $(I - \lambda\Pi_b)$ is invertible in L^{p_o} for all $-1 \leq \lambda \leq 1$.

Proof. If $|\Delta_k b| < 1 - \epsilon$ and $(I - \Pi_b)$ has a bounded inverse in L^p then, by Theorem 4.0.1, b is of RH_p^d-type $\iff$ the corresponding ω is doubling and in RH_p^d (by (ii) and (iv)) $\Rightarrow$ there exists $p_o > 1$ such that $\omega_\lambda \in RH_{p_o}^d$ for all $-1 \leq \lambda \leq 1$ (by Theorem 4.0.4), and ω_λ is certainly doubling (by Lemma 4.0.3) $\iff \lambda b$ is of RH_p^d-type and $|\Delta_k \lambda b| < 1 - \epsilon$, and using once more Theorem 4.0.1 we conclude that $(I - \lambda\Pi_b)$ is invertible in L^{p_o} for all $-1 \leq \lambda \leq 1$. □

Proof. [proof of Theorem 4.0.4] Given a doubling weight $\omega \in RH_p^d$ then $\omega \in A_\infty^d$ (by property (b)). By Lemma 4.0.3 it is also true that for $-1 \leq \lambda \leq 1$, ω_λ are doubling A_∞^d weights, with A_∞ constants uniformly bounded. That implies (see Remark 1 in page 3) the existence of $p_o > 1$ so that $\omega_\lambda \in RH_p^d$ for all $p > p_o$ and for all $-1 \leq \lambda \leq 1$. □

5. Counterexample

Theorem 5.0.6. There exist a doubling dyadic weight ω on $[0,1]$, $1<p<\infty$, and $0<\lambda<1$, such that $\omega \in RH_p^d$ but $\omega_\lambda \notin RH_p^d$.

The proof of this theorem will follow easily from the next two lemmas.

Lemma 5.0.7. There exists a one-parameter family of weights ω^t, $1/2 \leq t \leq 1$, with the following properties:

1) $\int_0^1 \omega^t = 1$ for all $1/2 \leq t \leq 1$,
2) ω^t is doubling for all $1/2 \leq t \leq 1$,
3) $\omega^t \in RH_p^d([0,1])$ if and only if $1 < p < p_t$, where $p_t = \ln 8/\ln[8t^2(t-1)]$ for $1/2 \leq t < (1+\sqrt{5})/4$, and $p_t = \infty$ for $(1+\sqrt{5})/4 \leq t \leq 1$.

The weights ω^t constructed in the previous lemma have the property that their structure is preserved under the operation $(\omega^t)_\lambda$, in particular we can keep track of the RH_p^d classes that they belong to. More precisely:

Lemma 5.0.8. $(\omega^t)_\lambda = \omega^{t_\lambda}$, where $t_\lambda = 1/2 + \lambda(t - 1/2)$, for all $1/2 \leq t \leq 1$.

Proof. [Proof of Theorem 5.0.6] Choose $(1+\sqrt{5})/4 \leq t_o \leq 1$ (eg. $t_o = 7/8$). Set $\omega = \omega^{t_o}$. By Lemma 5.0.7, ω is doubling and $\omega \in RH_p^d$ for all $1 < p < \infty$. In particular, $\omega \in RH_{p_o}^d$ for $p_o = p_{(t_o)_\lambda} > 1$, where $(t_o)_\lambda = 1/2 + \lambda(t_o - 1/2) > 1/2$, and we choose $\lambda > 0$ such that

$$(t_o)_\lambda < (1+\sqrt{5})/4$$

(for $t_o = 7/8$, and $(t_o)_\lambda = 2/3$ we get $\lambda = 4/9 < 1$).

Combining lemmas 5.0.8 and 5.0.7, we get that $\omega_\lambda = (\omega^{t_o})_\lambda = \omega^{(t_o)_\lambda}$ is doubling and belongs to RH_p^d only for $1 < p < p_{(t_o)_\lambda}$; in particular it does not belong to $RH_{p_o}^d$ for $p_o = p_{(t_o)_\lambda}$ (in our example $p_o = p_{2/3} = \ln 8/\ln(32/27) > 1$). □

Proof. [Proof of Lemma 5.0.7]

Fix $1/2 \leq t \leq 1$.

Let $I_k = (2^{-k}, 2^{-k+1}]$ for $k = 1, 2, \ldots$. Clearly $I_o = (0,1] = \cup_{k=1}^\infty I_k$. Define the step function

$$\omega^t(x) = \sum_{k=1}^\infty c_k(t)\chi_{I_k}(x),$$

where if we let $s = 1 - t$, then

$$c_k(t) = \begin{cases} 2^k(t^2 s)^n s & k = 3n+1 \\ 2^k(t^2 s)^n ts & k = 3n+2 \\ 2^k(t^2 s)^n t^3 & k = 3n+3 \end{cases}$$

Remark: The numbers t and s represent the proportion of the mass of ω on a given interval I_k^* that we have distributed among its children I_{k+1}, I_{k+1}^*; where $I_k^* = \cup_{j>k} I_j$ (the sibling of I_k).

1) Just computing and since $s + t = 1$, we get

$$\int_0^1 \omega^t = \sum_{k=1}^\infty c_k(t)|I_k| = (s + ts + t^3)\sum_{n=0}^\infty (t^2 s)^n = \frac{s + ts + t^3}{1 - t^2 s} = 1$$

2) We want to show that ω^t is a dyadic doubling weight. We must check that the mass in any dyadic interval I is comparable with that of its parent $\tilde{I}$, i.e. $\omega^t(\tilde{I}) \leq C\omega^t(I)$, for all $I \in \mathcal{D}(I_o)$. It is enough to consider only those intervals $\tilde{I}_k = I_k \cup I_k^*$. Because of the scale invariance it is enough to consider only the case $k=1$, $\tilde{I}_1 = I_o = I_1 \cup I_1^*$, $I_1^* = \cup_{j\geq 2} I_j$. By 1) $\omega^t(\tilde{I}_1) = 1$, by definition $\omega^t(I_1) = s = 1-t \leq 1/2$. It follows that $\omega^t(I_1^*) = t \geq 1/2$. Clearly $\omega^t(\tilde{I}) \leq s^{-1}\omega(I)$.
3) We want to show that $\omega^t \in RH_p^d$ for $1 < p < p_t$. Scaling once more, it is enough to check that for $p < p_t$
$$\int_0^1 (\omega^t)^p \leq C\left(\int_0^1 \omega^t\right)^p = C.$$
Now $(\omega^t)^p(x) = \sum_{k+1}^{\infty} c_k^p \chi_{I_k}(x)$, hence
$$\begin{aligned}\int_0^1 (\omega^t)^p(x)dx &= \sum_{k+1}^{\infty} c_k^p 2^{-k} \\ &= \left((2s)^p + (2^2ts)^p + (2^3t^3)^p\right)\sum_{n=0}^{\infty}\left[\frac{(2^3t^2s)^p}{2^3}\right]^n.\end{aligned}$$
This series converges if and only if
$$\frac{(2^3t^2s)^p}{2^3} < 1. \tag{9}$$
Recall that $s = 1-t$. Set $f(t) = 8t^2(1-t)$. It is a straightforward calculation to check that for $(1+\sqrt{5})/4 \leq t \leq 1$ then $f(t) \leq 1$. Hence in this range of t's (9) holds for every $1 < p$, i.e. $p_t = \infty$. For $1/2 \leq t < (1+\sqrt{5})/4$ we have that $f(t) > 1$. Therefore (9) holds only if and only if $p < p_t = \ln 8/\ln(8t^2(1-t))$.

This finishes the proof of the lemma. □

Proof. [Proof of Lemma 5.0.8] We can use the results in §3 to write:
$$\begin{aligned}\omega^t &= \prod_{k=0}^{\infty}(1+\Delta_k b^t), \\ \omega_\lambda^t &= \prod_{k=0}^{\infty}(1+\lambda\Delta_k b^t).\end{aligned}$$
We can find explicitly b^t. Recall that
$$\Delta_k b^t = \frac{\Delta_k\omega^t}{E_k\omega^t} = \frac{E_{k+1}\omega^t - E_k\omega^t}{E_k\omega^t}.$$
By the definition of ω^t, it is clear that $\Delta_k b^t(x)$ is not 0 only for those $x \in I_k^* = \cup_{j>k} I_j$, and in that case, $|\Delta_k b^t(x)| = t-s$ (by the remark in the proof of Lemma 5.0.7). This implies that for $0 < \lambda < 1$
$$|\lambda\Delta_k b^t(x)| = \begin{cases} \lambda(t-s) & x \in I_k^* \\ 0 & \text{otherwise}\end{cases}$$
We can write $\lambda(t-s) = t_\lambda - s_\lambda$ where $t_\lambda = 1/2 + \lambda(t-1/2)$ and $s_\lambda = 1/2 - \lambda(t-1/2)$; clearly $t_\lambda + s_\lambda = 1$. The structure of the weight ω_λ^t is exactly the same as the structure of the initial weight ω^t, except that we now replace t by t_λ. □

More refined versions of this counterexample will appear elsewhere. A slightly more delicate construction will allow us to construct examples where the index p_o in Theorem 5.0.6 can be made as close to one as we want (in our example the worst $p_o = p_{2/3}$).

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