Divergence-Free Multiwavelets

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Abstract. In this paper we construct $\mathbb{R}^n$-valued biorthogonal, compactly supported multiwavelet families such that one of the biorthogonal pairs consists of divergence-free vector wavelets. The construction is based largely on Lemarié's idea of multiresolution analyses intertwined by differentiation. We show that this technique extends nontrivially to multiwavelets via Strela's two-scale transform. An example based on the Donovan-Geronimo-Hardin-Massopust (DGHM) multiwavelets is given.

§1. Introduction

The study of divergence-free wavelets originated in the early 1990s with two completely different constructions: one due to Battle and Federbush [1], the other due to Lemarié [7]. The Battle-Federbush machine built interscale orthogonal wavelets. Later refinements gave orthogonal wavelets in dimensions 1, 2, 4, 8. None of these wavelet families have compact support. The Lemarié wavelets, on the other hand, are biorthogonal and compactly supported. Only one of the two biorthogonal families is divergence-free. His construction hinged on an important realization that multiresolution analyses (MRA's) could be related to one-another by differentiation, which is why biorthogonality is required (cf. [6]).

Federbush [3] used divergence-free wavelets to study uniqueness for the Navier-Stokes equations (NSE). Numerical analysis of NSE using variations of Lemarié wavelets has been implemented by K. Urban, et al. [9]. Other wavelet-Galerkin approaches to NSE do not use divergence-free wavelets, cf. Glowinski [5].

It is our thesis that, if wavelets are to have an impact on the numerical analysis of NSE, the wavelets should be divergence-free; in addition, they should have the best possible tradeoff between localization and approximation. This is a hypothesis at this stage: testing it will be the subject of future efforts. The burden of the present paper is to show how to construct adequate wavelets. The procedure is simple enough. We follow Lemarié's method for the most part, but we do so using multiwavelets. The tool for relating biorthogonal

Approximation Theory IX, Volume 2: Computational Aspects
Charles K. Chui and Larry L. Schumaker (eds.), pp. 161–168.

ISBN 0-8265-1326-3.

multi-MRA's (MMRA's) by differentiation is a result in Strela's thesis [8]. We build in the divergence-free property starting from the biorthogonal MMRA's. The biorthogonal filters associated to the DGHM [2] wavelets are recorded.

§2. MMRA's and Biorthogonality

First we need some handy, though abusive, notation. We will think of a real-valued multi-scaling function $\Phi(x)$ as a vector $[\phi^1(x)\dots\phi^m(x)] : \mathbb{R} \to \mathbb{R}^m$. Translation and dilation are performed componentwise. Thus $\Phi_{jk}(x) = [\phi^1_{jk}, \dots, \phi^m_{jk}]$, $\phi^i_{jk} = 2^{j/2}\phi^i(2^j x - k)$. When we write a wavelet or multiresolution expansion, for example, when we write the projection of $f : \mathbb{R} \to \mathbb{R}$ onto V_0, the l^2-closed linear span of the translates $\Phi_k(x) = \Phi(x-k)$ of Φ, as

$$P_{V_0} f(x) = \sum_{k\in\mathbb{Z}} \langle f, \Phi_k\rangle \Phi_k(x), \tag{1}$$

we really mean the integral of $f(y)$ against the kernel

$$K_0(x,y) = \sum_{k\in\mathbb{Z}} \Phi_k(x)\cdot\Phi_k(y), \tag{2}$$

where the dot product is taken in $\mathbb{R}^m$. In particular, we will always interpret quantities like the right side of (1) as being scalar-valued.

Since much of our analysis is done at the filter level, we recall here the basic properties of scaling filters. The fact that Φ is an $\mathbb{R}^m$-valued scaling function means that there is a sequence $\{C_k\} \subset \mathcal{M}_{m\times m}(\mathbb{R})$ such that

$$\Phi(x) = \sum_{k\in\mathbb{Z}} C_k \Phi(2x-k) \tag{3}$$

which, upon taking Fourier transforms of both sides, can be rewritten

$$\hat\Phi(2\xi) = \frac{1}{2}\sum_k C_k e^{-2\pi i k\xi}\hat\Phi(\xi) = H(\xi)\hat\Phi(\xi). \tag{4}$$

It will be convenient to make the substitution $z = e^{-2\pi i\xi}$ so that the low pass filter $H(z) = H(\xi)$ is a matrix-valued Laurent series, or Laurent polynomial in the compactly supported case. If $H^*(\xi)$ denotes the conjugate transpose of $H(\xi)$ then, in terms of z, we have $H^*(z) = \sum_k C_k^* z^{-k}$ where C_k^* is the transpose of C_k. We will denote by the high pass filter $F(\xi)$ a matrix valued trigonometric series such that $\hat\Psi(2\xi) = F(\xi)\hat\Phi(\xi)$ defines a multiwavelet Ψ. For the sake of brevity we will not be pedantic about what conditions on H are needed to guarantee convergence/regularity of Φ, nor about issues in the design of F in terms of H; there is ample description in the literature. In any case, we shall assume throughout that Φ has enough regularity that convergence of both sides of any identity can be justified. Instead, we will focus on the biorthogonality conditions. In what follows, we will consider biorthogonal MMRA's having scaling functions Φ_+, Φ_- that come from smoothing or roughening the scaling function Φ from a fixed orthogonal MMRA. One could equally well derive new biorthogonal MMRA's starting from fixed biorthogonal MMRA's. The *conditions of biorthogonality* for the derived MMRA's are then (cf. [8])

$$\begin{aligned} H_+(z)H_-^*(z) + H_+(-z)H_-^*(-z) &= I \\ F_+(z)F_-^*(z) + F_+(-z)F_-^*(-z) &= I \\ H_+(z)F_-^*(z) + H_+(-z)F_-^*(-z) &= 0 \\ H_-(z)F_+^*(z) + H_-(-z)F_+^*(-z) &= 0. \end{aligned} \tag{5}$$

The Fourier transforms of the biorthogonal scaling functions and wavelets, respectively, are then given by

$$\hat{\Phi}_\pm(2\xi) = H_\pm(\xi)\hat{\Phi}_\pm(\xi); \qquad \hat{\Psi}_\pm(2\xi) = F_\pm(\xi)\hat{\Phi}_\pm(\xi). \tag{6}$$

In the case $H_+ = H_-$ and $F_+ = F_-$ these are the conditions of orthogonality.

§3. Two-Scale Transform and Differentiation

To relate a new scaling filter to an old one, Strela invented the two-scale transform. This means that one starts with a transition matrix $M(z) = M(\xi)$, $z = e^{-2\pi i\xi}$ and defines

$$\begin{aligned} H(z) &= \frac{1}{2}M(z^2)H_-(z)M^{-1}(z) \\ H_+(z) &= \frac{1}{2}M^*(z^2)H(z)M^{*\,-1}(z) \end{aligned} \tag{7}$$

provided M^{-1} exists. If H satisfies the orthogonality condition $HH^*(z) + HH^*(-z) = I$, then since

$$\begin{aligned} H_+H_-^*(z) &= \left[\frac{1}{2}M^*(z^2)H(z)M^{*\,-1}(z)\right][2M^{-1}(z^2)H(z)M(z)]^* \\ &= M^*(z^2)HH^*(z)M^{*\,-1}(z^2), \end{aligned}$$

it follows that

$$H_+H_-^*(z) + H_+H_-^*(-z) = I.$$

Therefore, H_+ and H_- will give rise to a new pair of biorthogonal MMRA's under suitable convergence conditions. These and the role of differentiation are provided by the following proposition due to Strela [8]:

Proposition 1. *Let Φ be a scaling vector with filter H providing approximation order at least one. If $M(\xi)$ is invertible for all $\xi \neq 0$ ($z \neq 1$), $M(0)$ has left eigenvector $\hat{\Phi}(0)$ corresponding to the simple eigenvalue $\lambda(0) = 0$, and $\frac{d}{dz}\det M(z)|_{z=1} \neq 0$ then*

$$\hat{\Phi}_+(\xi) = \frac{1}{2\pi i\xi}M^*(\xi)\hat{\Phi}(\xi)$$

solves the scaling equation

$$\hat{\Phi}_+(2\xi) = H_+(\xi)\hat{\Phi}_+(\xi)$$

with H_+ *defined by* (7). *Similarly,* $2\pi i\xi\hat{\Phi}(\xi) = M(\xi)\hat{\Phi}_-(\xi)$.

Strictly speaking, some technical regularity conditions, which are met in the example of Section 6, should be imposed to insure convergence of $\Phi_\pm$. Alternatively, one can take convergence in the sense of distributions. Since $M^*(\xi)$ is a matrix trigonometric series, its inverse Fourier transform is a matrix T^* of generalized translation operators. The equations relating $\hat{\Phi}_\pm$ to $\hat{\Phi}$ become $D\Phi_+ = -T^*\Phi$ and $D\Phi = -T\Phi_-$ where $D = d/dx$ is taken componentwise. Then $D\Phi_+ \in V_0$ provided the coefficients of T^* are in $l^2(\mathbb{Z})$.

Given these scaling functions, how should the wavelets be designed? Typically one can choose the wavelets to satisfy as many desirable properties, such as symmetry or short support, as possible. In the present setting, the simplest high-pass filters are

$$\begin{aligned} F(z) &= \frac{1}{2}F_-(z)M^{-1}(z) \\ F_+(z) &= \frac{1}{2}F(z)M^{*-1}(z). \end{aligned} \tag{8}$$

Lemma 2. *If the filters* H, F *satisfy the conditions of orthogonality then the filters* $H_\pm, F_\pm$ *satisfy the conditions* (5) *of biorthogonality.*

This is easy to see. For example,

$$F_+(z)F_-^*(z) = \left[\frac{1}{2}F(z)M^{*-1}(z)\right][2F(z)M(z)]^* = F(z)F^*(z)$$

so that, if $FF^*(z) + FF^*(-z) = I$ then $F_+F_-^*(z) + F_+F_-^*(-z) = I$ as well. The corresponding multiwavelets are related by differentiation in the simplest way possible.

Lemma 3. *The multiwavelets* $\Psi_\pm$ *satisfy*

$$\begin{aligned} D\Psi_+ &= -\Psi \\ D\Psi &= -\Psi_-. \end{aligned}$$

Proof: In terms of Fourier transforms we have:

$$\begin{aligned} \hat{\Psi}_+(2\xi) = F_+(\xi)\hat{\Phi}_+(\xi) &= \frac{1}{2\pi i\xi}F_+(\xi)M^*(\xi)\hat{\Phi}(\xi) \\ &= \frac{1}{2\pi i\xi}\frac{1}{2}F(\xi)M^{*-1}(\xi)M^*(\xi)F^{-1}(\xi)\hat{\Psi}(2\xi) = \frac{1}{2\pi i\xi}\frac{1}{2}\hat{\Psi}(2\xi). \end{aligned}$$

Substituting ξ for 2ξ and inverse Fourier transforming shows that $D\Psi_+ = -\Psi$. The proof that $D\Psi = -\Psi_-$ is similar. □

§4. Differentiation and Multiresolution Expansions

To check that differentiation intertwines wavelet expansions let $\Phi, \Phi_\pm$ and $\Psi, \Psi_\pm$ be related by (7) and (8). Let V_0 and W_0 be the multiresolution and wavelet spaces spanned by the translates of Φ and Ψ, and let $V_{0,+}$ denote the span of translates of Φ_+, and $W_{0,+}$ the span of $\Psi_{+,k}$ in the sense of (1). The MRA properties imply that $V_{1,+} = V_{0,+} + W_{0,+}$, respecting biorthogonality with $V_{1,-} = V_{0,-} + W_{0,-}$, and replicable at any scale. In one sense, it is obvious that the MRA expansions commute with differentiation.

Lemma 4. *If $f \in V_{0,+}$ and $g \in W_{0,+}$ then $Df \in V_0$, $Dg \in W_0$, and*

$$\begin{aligned} Df &= \sum_{k\in\mathbb{Z}} \langle f, \Phi_{-,k}\rangle D\Phi_{+,k} = \sum_{k\in\mathbb{Z}} \langle Df, \Phi_k\rangle \Phi_k \\ Dg &= \sum_{k\in\mathbb{Z}} \langle g, \Psi_{-,k}\rangle D\Psi_{+,k} = \sum_{k\in\mathbb{Z}} \langle Dg, \Psi_k\rangle \Psi_k. \end{aligned}$$

As always, we assume that convergence is no problem. For example, if Φ has compact support and any Hölder regularity, then both sides of both identities will converge absolutely. Though the identities are obvious, the fact that they are related through summation by parts will be important below. The W_0 case is trivial, so we demonstrate the V_0 case. Since $D\Phi_+ = -T^*\Phi$, and the same holds for the translates Φ_k, we have

$$\begin{aligned} D\sum_k \langle f, \Phi_{-,k}\rangle \Phi_{+,k} &= \sum_k \langle f, \Phi_{-,k}\rangle D\Phi_{+,k} = -\sum_k \langle f, \Phi_{-,k}\rangle T^*\Phi_k \\ &= \sum_k \langle f, T\Phi_{-,k}\rangle \Phi_k = \sum_k \langle Df, \Phi_k\rangle \Phi_k. \end{aligned}$$

The third equation is nothing but summation by parts and change of variables in the integrals defining the scaling coefficients.

§5. Form-Valued Wavelets

For multivariate expansions we need to take tensor-products of univariate wavelets. We follow the construction in [4] (cf. [9]) now to create vector-valued biorthogonal wavelet families. Then we will build in the divergence-free condition. Even if one is solely interested in $\mathbb{R}^n$-valued wavelets, the project is carried out most naturally at the level of r-forms.

To do this we need some notation. As usual, $\epsilon \in E^*$ means that $\epsilon = (\epsilon_1, \dots, \epsilon_n) \in \{0,1\}^n \setminus (0,0)$. Set $i_\epsilon = \min\{i : \epsilon_i = 1\}$. Functions $f = \sum_{\alpha\in\mathcal{P}_r} f_\alpha e_\alpha$ where e_α is a basis for the r-th exterior product of $\mathbb{R}^n$ indexed by subsets $\alpha \in \mathcal{P}_r = \{1 \le i_1 < i_2 < \dots < i_r \le n\}$ are *r forms.* If $j \in \alpha$ then $\alpha \setminus j \in \mathcal{P}_{r-1}$ is obtained by deleting j from α. For any fixed $\epsilon \in E^*$, set

$$\mathcal{T}_r = \mathcal{T}_r^\epsilon = \{\alpha \in \mathcal{P}_r : i_\epsilon \notin \alpha\}; \qquad \mathcal{N}_r = \mathcal{P}_r \setminus \mathcal{T}_r.$$

The decomposition $f = \sum_{\alpha\in\mathcal{T}_r} f_\alpha e_\alpha + \sum_{\alpha\in\mathcal{N}_r} f_\alpha e_\alpha = f_\mathcal{T} + f_\mathcal{N}$ is analogous to the Fourier decomposition of any form f into its 'tangential' and 'normal' components f_T and f_N: there f_T is the 'divergence-free' component; for us it will be a modification of $f_{\mathcal{T}}$.

To get biorthogonal bases for $L^2(\mathbb{R}^n)$ we use tensor products. In each component we use either the original basis generated by Ψ or the derived bases generated by $\Psi_\pm$, depending on the *signature* (ϵ, α). Precisely, define

$$\begin{aligned}\Gamma_{\epsilon,\alpha}(x) &= \prod_{j\in\alpha}(\epsilon_j\Psi_+(x_j) + (1-\epsilon_j)\Phi_+(x_j))\prod_{j\notin\alpha}(\epsilon_j\Psi(x_j) + (1-\epsilon_j)\Phi(x_j))\\ \Theta_{\epsilon,\alpha}(x) &= \prod_{j\in\alpha}(\epsilon_j\Psi_-(x_j) + (1-\epsilon_j)\Phi_-(x_j))\prod_{j\notin\alpha}(\epsilon_j\Psi(x_j) + (1-\epsilon_j)\Phi(x_j)).\end{aligned} \tag{9}$$

These functions, along with their translates and dilates, are biorthogonal to one another. Furthermore, for each fixed $\alpha \in \mathcal{P}_r$, the families $\{\Gamma_{\epsilon,\alpha}, \Theta_{\epsilon,\alpha}\}_{\epsilon\in E^*}$ give rise to biorthogonal bases for $L^2(\mathbb{R}^n)$.

To build bases for $L^2(\mathbb{R}^n, \Lambda_r(\mathbb{R}^n))$ we simply set

$$\Gamma^{\epsilon,\alpha} = \Gamma_{\epsilon,\alpha}e_\alpha; \qquad \Theta^{\epsilon,\alpha} = \Theta_{\epsilon,\alpha}e_\alpha.$$

Since $\{e_\alpha\}_{\alpha\in\mathcal{P}_r}$ forms an orthonormal basis for the Hilbert space $\Lambda_r(\mathbb{R}^n)$, the translates and dilates of $\{\Gamma^{\epsilon,\alpha}, \Theta^{\epsilon,\alpha}\}_{\epsilon\in E^*,\alpha\in\mathcal{P}_r}$ form biorthogonal bases for $L^2(\mathbb{R}^n, \Lambda_r(\mathbb{R}^n))$. It remains to construct a basis for the Hodge projection onto the 'divergence-free' r-forms. This is where $\mathcal{T}_r$ and $\mathcal{N}_r$ come into play.

For each fixed ϵ and $\alpha \in \mathcal{T}_r^\epsilon = \mathcal{T}_r$, set $a = \alpha \cup i_\epsilon$ and set $\Psi^{\epsilon,\alpha} = -\mathrm{sgn}(i_\epsilon, a)d^*\Gamma^{\epsilon,a}$. The deRham operator $d^* = \sum_{i=1}^n \mu_i^*\partial/\partial x_i$. On vector fields $(r = 1)$ d^* is the divergence operator. Here, μ_i^* is the interior multiplication operator $\mu_i^* e_\alpha = \mathrm{sgn}(i,\alpha)e_{\alpha\setminus i}$ where $\mathrm{sgn}(i,\alpha)$ is 0 if $i \notin \alpha$ and ± 1 depending on whether i marks an odd/even place when the elements of α are listed in increasing order. One has

$$\Psi^{\epsilon,\alpha} = \Gamma^{\epsilon,\alpha} + \mathrm{sgn}(i_\epsilon, a)\sum_{j\in\alpha}\mathrm{sgn}(j,a)\Gamma^{\epsilon,a\setminus j}. \tag{10}$$

The pairs $\{(\Psi^{\epsilon,\alpha}, \Theta^{\epsilon',\alpha'})\}_{\alpha\in\mathcal{T}_r}$ are thus biorthogonal because $(\Gamma^{\epsilon,\alpha}, \Theta^{\epsilon',\alpha'})$ are and if $\alpha \in \mathcal{T}_r$ and $j \in \alpha$ then $a \setminus j \in \mathcal{N}_r$.

Since the distributional image of d^* on $r+1$-forms equals the distributional kernel of d^* on r-forms, the $\Psi^{\epsilon,\alpha}$ $(\alpha \in \mathcal{T}_r)$ are divergence-free in the sense that $d^*\Psi^{\epsilon,\alpha} = 0$. Therefore, the $\Psi^{\epsilon,\alpha}$ will form a basis for the divergence-free subspace of $L^2(\mathbb{R}^n, \Lambda_r)$ provided they are complete for this subspace. In turn, it suffices that the expansion of any divergence-free f in terms of $\{(\Psi^{\epsilon,\alpha}, \Theta^{\epsilon,\alpha})\}_{\alpha\in\mathcal{T}_r}$ agrees with its expansion in terms of $\{(\Gamma^{\epsilon,\beta}, \Theta^{\epsilon,\beta})\}_{\beta\in\mathcal{P}_r}$. By rescaling, this is true if both expansions agree at unit scale when f is in the kernel of d^*. First, because of Lemma 4,

$$\mu_{i_\epsilon}^*\frac{\partial}{\partial x_{i_\epsilon}}\Big(\sum_{\beta\in\mathcal{P}_r}\sum_{k\in\mathbb{Z}^n}\langle f, \Theta_k^{\epsilon,\beta}\rangle\Gamma_k^{\epsilon,\beta}\Big) = \sum_{\delta\in\mathcal{T}_{r-1}}\sum_k\Big\langle\mu_{i_\epsilon}^*\frac{\partial}{\partial x_{i_\epsilon}}f, \Theta_k^{\epsilon,\delta}\Big\rangle\Gamma_k^{\epsilon,\delta}. \tag{11}$$

On the other hand, if $\alpha \in \mathcal{T}_r$ then

$$\sum_k \langle f, \Theta_k^{\epsilon,\alpha}\rangle \Psi_k^{\epsilon,\alpha} = \sum_k \langle f, \Theta_k^{\epsilon,\alpha}\rangle \Gamma_k^{\epsilon,\alpha} \pm \sum_{j\in\alpha} \operatorname{sgn}(j,a) \sum_k \langle \mu_j^* \frac{\partial}{\partial x_j} f, \Theta_k^{\epsilon,\alpha\backslash j}\rangle \Gamma_k^{\epsilon,a\backslash j}.$$

Consequently we obtain

$$\mu_{i_\epsilon}^* \frac{\partial}{\partial x_{i_\epsilon}} \Big(\sum_{k\in\mathbb{Z}^n} \langle f, \Theta_k^{\epsilon,\alpha}\rangle \Psi_k^{\epsilon,\alpha} \Big) = -\sum_{j\in\alpha}\sum_k \Big\langle \mu_j^* \frac{\partial}{\partial x_j} f, \Theta_k^{\epsilon,\alpha\backslash j} \Big\rangle \Gamma_k^{\epsilon,a\backslash j}. \qquad (12)$$

Summing (12) over all α and comparing to (11) proves:

Theorem 5. *If $f \in L^2(\mathbb{R}^n, \Lambda_r(\mathbb{R}^n))$ satisfies $d^* f = 0$ then, for $\epsilon \in E^*$,*

$$\sum_{\beta\in\mathcal{P}_r} \sum_{k\in\mathbb{Z}^n} \langle f, \Theta_k^{\epsilon,\beta}\rangle \Gamma_k^{\epsilon,\beta} = \sum_{\alpha\in\mathcal{T}_r} \sum_{k\in\mathbb{Z}^n} \langle f, \Theta_k^{\epsilon,\alpha}\rangle \Psi_k^{\epsilon,\alpha}.$$

Similar identities hold for every scale. Thus the translates and dilates of $\Psi^{\epsilon,\alpha}$ ($\epsilon \in E^*, \alpha \in \mathcal{T}_r$) form a basis for the divergence free subspace.

§6. The Case of DGHM Multiwavelets

The scaling filter $H(z)$ in (4) for the DGHM orthogonal multiwavelets is

$$H(z) = \frac{1}{20} \begin{bmatrix} 6(1+z) & 8\sqrt{2} \\ (-1+9z+9z^2-z^3)/\sqrt{2} & -3+10z-3z^2 \end{bmatrix}.$$

It is easily checked that, together with the high pass filter

$$F(z) = \frac{1}{20} \begin{bmatrix} (-1+9z+9z^2-z^3)/\sqrt{2} & -3-10z-3z^2 \\ (-1+9z-9z^2+z^3) & 6(z^2-1)/\sqrt{2} \end{bmatrix},$$

the conditions of orthogonality hold. The wavelets give approximation order two.

Now it is a simple matter [8] to check that the matrix

$$M(z) = \begin{bmatrix} 0 & \sqrt{2} \\ 1-z & -1-z \end{bmatrix}$$

satisfies the hypotheses of Proposition 1 for H. Then by (7), $H_+(z)$ is

$$\frac{1}{80z^2} \begin{bmatrix} z-1+40z^2+40z^3+z^4-z^5 & (1-z)(1+z)^2(1-10z+z^2) \\ (z-1)(1-26z^2+z^4) & 1-9z+4z^2+4z^3-9z^4+z^5 \end{bmatrix},$$

whereas

$$H_-(z) = \frac{1}{10} \begin{bmatrix} 5(1+z) & 0 \\ 8(1-z) & -2(1+z) \end{bmatrix}.$$

The corresponding high pass filters are

$$F_+(z) = \frac{1}{80} \begin{bmatrix} 1-z+z^2-z^3 & -1+9z+9z^2-z^3 \\ \sqrt{2}(1-z-z^2+z^3) & \sqrt{2}(-1+9z-9z^2+z^3) \end{bmatrix},$$

whereas

$$F_-(z) = \frac{1}{10} \begin{bmatrix} (-1+z)(3+10z+3z^2) & 2(1+z)(1+10z+z^2) \\ -3(1-z^2)(1-z) & 2\sqrt{2}(1-z)(1+7z+z^2) \end{bmatrix}.$$

It is simple to check that the conditions of biorthogonality are met for these filters, so that the corresponding wavelets form biorthogonal bases that are related by differentiation.

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