PARAEXPONENTIALS, MUCKENHOUPT WEIGHTS, AND RESOLVENTS OF PARAPRODUCTS

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ABSTRACT. We analyze the stability of Muckenhoupt's $\mathbf{RH}_p^d$ and $\mathbf{A}_p^d$ classes of weights under a nonlinear operation, the λ-operation. We prove that the dyadic doubling reverse Hölder classes $\mathbf{RH}_p^d$ are not preserved under the λ-operation, but the dyadic doubling A_p classes $\mathbf{A}_p^d$ are preserved for $0 \le \lambda \le 1$. We give an application to the structure of resolvent sets of dyadic paraproduct operators.

1. Introduction

The Muckenhoupt classes of weights consist of positive locally integrable functions satisfying certain integrability conditions on intervals; see Section 2. They arise, for instance, in connection with the boundedness of the Hilbert transform and maximal function operators on L^p spaces.

Weights in the Muckenhoupt classes are often represented as exponentials of functions in $\mathbf{BMO}$; see [GC-RF]. For weights $\omega(x)$ defined on an interval J, with mean value one on J, a different representation, sometimes called the *paraexponential*, was introduced in [FKP]. The correspondence is realized by an infinite product, namely:

$$\omega(x) = \prod_{I \in \mathcal{D}(J)} \big(1 + b_I h_I(x)\big), \qquad b_I = \frac{\langle \omega, h_I \rangle}{m_I \omega}; \tag{1.1}$$

where $\mathcal{D}(J)$ denotes the intervals in the dyadic decomposition of the interval J, $\{h_I\}_{I \in \mathcal{D}(J)}$ are Haar functions, $\langle \cdot, \cdot \rangle$ is the scalar product in L^2, and $m_I\omega$ is the mean value of ω on I.

It was proved in [FKP] that if there is an $\varepsilon > 0$ such that $|b_I h_I| < 1 - \varepsilon$ for all $I \in \mathcal{D}(J)$, then the partial products converge to a weight in dyadic doubling $\mathbf{A}_\infty^d$ if and only if $b = \sum_{I \in \mathcal{D}(J)} b_I h_I$ is a function of dyadic bounded mean oscillation, $\mathbf{BMO}^d$. A larger dictionary relating properties of ω and b can be found in [B]. In particular the dyadic doubling Muckenhoupt $\mathbf{RH}_p^d$ and $\mathbf{A}_p^d$ classes can be characterized by summation conditions on b.

We will consider only weights ω defined on an interval J, with mean value one on J. For such a weight we have, at least formally, the product representation (1.1). We are interested in the effect of multiplying the coefficient b_I in each factor of the

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product by the same number λ. Define

$$\omega_\lambda(x) = \prod_{I\in\mathcal{D}(J)} \big(1+\lambda b_I h_I(x)\big). \tag{1.2}$$

We call the mapping that sends $\omega \mapsto \omega_\lambda$ the λ-*operation*.

To guarantee the convergence of the products we will only consider the case $-1 \le \lambda \le 1$. If ω is a weight in $\mathbf{A}^d_\infty$ and $-1 \le \lambda \le 1$, then ω_λ is also in $\mathbf{A}^d_\infty$; see [FKP].

The λ-operation seems, at first sight, very similar to taking a λ power of the weight, ω^λ. The first two terms in the Taylor expansion of $\omega^\lambda = e^{\lambda b}$ coincide with the first two terms obtained by expanding the infinite product in (1.2), but the third terms differ. Both the $\mathbf{RH}^d_p$ and the $\mathbf{A}^d_p$ classes are preserved under taking powers $\omega \mapsto \omega^\lambda$, for $0 \le \lambda \le 1$.

Question: Are the $\mathbf{RH}^d_p$ and $\mathbf{A}^d_p$ classes preserved under the λ-operation?

The answer is negative for the $\mathbf{RH}^d_p$ classes, and positive for the $\mathbf{A}^d_p$ classes.

Theorem 1.1. *For each $p > 1$, there exist a weight ω in $\mathbf{RH}^d_p$ and a number $\lambda \in (0,1)$ such that ω_λ is not in $\mathbf{RH}^d_p$.*

For $p > (1-\log 2)^{-1} > 1$, examples with positive λ's are given in [P2]. In this paper we give examples with positive λ's for all $p > 1$. Examples for all $p > 1$, with $-1 \le \lambda < 0$, are given in [P1].

A related result in [P2] is that if $\omega \in \mathbf{RH}^d_p$, then there is a $q > 1$ such that $\omega_\lambda \in \mathbf{RH}^d_q$ for all $-1 \le \lambda \le 1$.

Theorem 1.2. *Take any p and λ such that $1 \le p \le \infty$ and $0 \le \lambda \le 1$. If the weight ω is in $\mathbf{A}^d_p$, then the weight ω_λ is also in $\mathbf{A}^d_p$.*

However, the $\mathbf{A}^d_p$ spaces for $1 \le p < \infty$ are not preserved by the λ-operation for negative λ; this can be shown by examples like those in [P1].

The λ-operation appears naturally in the study of the resolvents of the dyadic paraproduct [P2]. The dyadic paraproduct is a bilinear operator that appears in different guises in harmonic analysis, often replacing the ordinary product [M, Ch, D]. The dyadic paraproduct π_b associated to a function b is defined by

$$\pi_b f(x) = \sum_{I\in\mathcal{D}(J)} m_I f\, b_I h_I(x), \tag{1.3}$$

where $m_I f$ is the mean value of f on the interval I, and $b_I = \langle b, h_I\rangle$. The paraproduct π_b is bounded on $L^p(J)$ if and only if $b \in \mathbf{BMO}^d$. The conditions on b that guarantee the existence of a bounded inverse of $(I-\lambda\pi_b)$ on $L^p_o(J) = \{f \in L^p(J) : \int_J f = 0\}$ are described in terms of properties of the weight ω_λ. For doubling weights, the conditions reduce to $\omega_\lambda \in \mathbf{RH}^d_p$; see [P1]. More precisely:

Theorem 1.3 (P)**.** *Let b be a function in $\mathbf{BMO}^d$, take $\varepsilon > 0$, and let λ be a real number such that $|\lambda b_I h_I| \le 1-\varepsilon$ for all $I \in \mathcal{D}(J)$, where $b_I = \langle b, h_I\rangle$. Then $(I-\lambda\pi_b)^{-1}$ exists and is bounded on $L^p_o(J)$ if and only if the weight $\omega_\lambda(x) = \prod_{I\in\mathcal{D}(J)}\big(1+\lambda b_I h_I(x)\big)$ is in $\mathbf{RH}^d_p$.*

Question: If $(I-\lambda\pi_b)^{-1}$ exists and is bounded on L^p_0 for $\lambda=1$, is the same true for all $\lambda\in[0,1]$?

Since the resolvent set of π_b is open, there are neighborhoods of $\lambda=0$ and $\lambda=1$ for which this holds. By the discussion above, the question is equivalent to asking whether the $\mathbf{RH}^d_p$ classes are preserved under the λ-operation. Theorem 1.1 shows that the answer is negative. Rephrasing the theorem we have:

Theorem 1.4. *For each $p>1$, there exist a number $\lambda\in(0,1)$, a function $b\in\mathbf{BMO}^d$, and a number $\varepsilon>0$ such that $|b_I h_I|\le 1-\varepsilon$ for all $I\in\mathcal{D}(J)$, the operator $(I-\pi_b)^{-1}$ exists and is a bounded operator on $L^p_o(J)$, but the operator $(I-\lambda\pi_b)$ is not invertible as an operator from $L^p_o(J)$ into $L^p_o(J)$.*

In Section 2 we give notation, definitions, and some preparatory lemmas. In Section 3 we prove Theorem 1.2, and in Section 4 we construct the examples which establish Theorem 1.1.

As customary, C denotes a constant that may change from line to line.

2. Preliminaries

In this section we introduce the dyadic intervals and the Haar basis; and we define dyadic $\mathbf{BMO}^d$ in terms of a Carleson condition. Next we define the dyadic doubling Muckenhoupt classes $\mathbf{A}^d_\infty$, $\mathbf{RH}^d_p$, and $\mathbf{A}^d_p$; and we recall some of their properties. We state the Fefferman-Kenig-Pipher Product Representation Theorem for $\mathbf{A}^d_\infty$ weights and Buckley's Theorem characterizing $\mathbf{A}^d_p$ weights via summation conditions. Finally we define the λ-operation, and give a convexity lemma used in the proof of Theorem 1.2.

Let $\mathcal{D}$ denote the family of all dyadic subintervals of $[0,1]$, in other words all intervals of the form $(j2^{-k},(j+1)2^{-k}]$, j, k integers, $0\le k$, $0\le j\le 2^k-1$. Given any interval J, $\mathcal{D}(J)$ denotes the family of dyadic subintervals of J. Given an interval J we denote its left and right halves respectively by J_l and J_r. An interval $\widetilde{I}$ is the *parent* of an interval I if I is $\widetilde{I}_l$ or $\widetilde{I}_r$.

The *Haar function* associated to an interval I is given by $h_I(x)=|I|^{-1/2}\big(\chi_{I_r}(x)-\chi_{I_l}(x)\big)$; here χ_I denotes the characteristic function of the interval I. The set of Haar functions indexed by $\mathcal{D}(J)$ forms a basis of $L^2_o(J)=\{f\in L^2(J):\int_J f=0\}$; see [H].

A locally integrable function b on $[0,1]$ is in the space of *dyadic bounded mean oscillation* $\mathbf{BMO}^d$ if there is a constant C such that $\int_J|b(x)-m_Jb|^2\,dx\le C\,|J|$ for all $J\in\mathcal{D}$, where $m_Jb=\frac{1}{|J|}\int_J b$. The function b is in $\mathbf{BMO}^d$ if and only if the *Carleson condition* on the Haar coefficients $b_I=\langle b,h_I\rangle$ of b holds: $\sum_{I\in\mathcal{D}(J)}b_I^2\le C\,|J|$ for all $J\in\mathcal{D}$, with a constant C independent of J; see [Ch, M].

2.1. Dyadic weights.

We consider weights defined on the interval $J_0=[0,1]$.

A *dyadic doubling weight* ω is a positive locally integrable function such that $\int_{\widetilde{I}}\omega\le C\int_I\omega$ for all intervals $I\in\mathcal{D}$, where $\widetilde{I}$ is the parent of I and C is a constant independent of I.

A weight ω is in the *dyadic doubling A_∞ class* $\mathbf{A}^d_\infty$ if ω is dyadic doubling, and there is a constant C such that $\frac{1}{|I|}\int_I\omega\le C\exp\left(\frac{1}{|I|}\int_I\log\omega\right)$, for all $I\in\mathcal{D}$.

A weight ω is in the *dyadic doubling reverse Hölder class* $\mathbf{RH}_p^d$, for $1<p<\infty$, if ω is dyadic doubling, and there is a constant C such that $\left(\frac{1}{|I|}\int_I \omega^p\right)^{1/p} \leq C\,\frac{1}{|I|}\int_I \omega$, for all $I\in\mathcal{D}$.

A weight ω is in the *dyadic doubling A_p class* $\mathbf{A}_p^d$, for $1<p<\infty$, if ω is dyadic doubling, and there is a constant C such that $\left(\frac{1}{|I|}\int_I \omega\right)\left(\frac{1}{|I|}\int_I \omega^{-1/(p-1)}\right)^{p-1} \leq C$ for all $I\in\mathcal{D}$.

A weight ω is in the *dyadic doubling A_1 class* $\mathbf{A}_1^d$ if ω is dyadic doubling, and there is a constant C such that $\frac{1}{|I|}\int_I \omega \leq C\,\omega(x)$ for a.e. $x\in I$, for all $I\in\mathcal{D}$.

The canonical examples are $\omega(x)=|x|^\alpha$. In this case $\omega\in\mathbf{A}_\infty^d$ if and only if $\alpha>-1$, $\omega\in\mathbf{RH}_p^d$ if and only if $\alpha>-1/p$, for $1<p<\infty$ $\omega\in\mathbf{A}_p^d$ if and only if $-1<\alpha<p-1$, and $w\in\mathbf{A}_1^d$ if and only if $-1<\alpha\leq 0$.

The class $\mathbf{A}_\infty^d$ is the union of the $\mathbf{RH}_p^d$ classes, and also of the $\mathbf{A}_p^d$ classes. The class $\mathbf{A}_1^d$ is strictly contained in the intersection of the $\mathbf{A}_p^d$ classes. More precisely:

$$\mathbf{A}_\infty^d=\bigcup_{p>1}\mathbf{RH}_p^d=\bigcup_{p>1}\mathbf{A}_p^d,\qquad \mathbf{A}_1^d\subset\bigcap_{p>1}\mathbf{A}_p^d. \tag{2.1}$$

These properties are known for weights that satisfy the conditions above for all intervals I; see [GC-RF]. In our case we consider only dyadic intervals, but we also assume that the weights are dyadic doubling, which yields (2.1); see [B]. An explicit example of a function in $\bigcap_{p>1}\mathbf{A}_p^d$ but not in $\mathbf{A}_1^d$ can be found in [JN].

To each weight $\omega\in\mathbf{A}_\infty^d$, with mean value one on J_0, we associate a function $b=b_\omega$ so that

$$\omega(x)=\prod_{I\in\mathcal{D}}\big(1+b_I h_I(x)\big),\qquad b_I=\frac{\langle\omega,h_I\rangle}{m_I\omega}, \tag{2.2}$$

and

$$b(x)=\sum_{I\in\mathcal{D}} b_I h_I(x).$$

Theorem 2.1 (R. Fefferman, Kenig, Pipher)**.** *Let $\{b_I\}_{I\in\mathcal{D}(J_0)}$ and $\varepsilon>0$ be given, with $|b_I h_I|\leq 1-\varepsilon$ for all $I\in\mathcal{D}$. Then the product* (2.2) *belongs to $\mathbf{A}_\infty^d$ if and only if $b\in\mathbf{BMO}^d$, that is, if and only if there is a constant C such that*

$$\sum_{I\in\mathcal{D}(J)} b_I^2\leq C\,|J|,\quad \forall\, J\in\mathcal{D}. \tag{2.3}$$

If so, the weight ω is a dyadic doubling weight.

This is proved in [FKP].

The dyadic $\mathbf{RH}_p^d$ and $\mathbf{A}_p^d$ classes can also be characterized by summation conditions. Part (a) of the next theorem, and related results, appear in [B].

Theorem 2.2. *Let ω be a dyadic doubling weight ω.*

(a) (Buckley) $\omega\in\mathbf{A}_p^d$, $1<p<\infty$, *if and only if there is a constant C such that*

$$\sum_{I\in\mathcal{D}(J)}\left(\frac{m_I\omega}{m_J\omega}\right)^{\frac{-1}{p-1}} b_I^2\leq C\,|J|,\qquad \forall\, J\in\mathcal{D}.$$

(b) $w \in \mathbf{A}_1^d$ *if and only if there is a constant* C *such that*

$$\frac{m_J\omega}{m_I\omega} \leq C, \qquad \forall\, I \in \mathcal{D}(J),\ \ \forall\, J \in \mathcal{D}.$$

Proof of (b): ($\Rightarrow$) Integrate over $I \in \mathcal{D}(J)$ in the definition of $\mathbf{A}_1^d$. ($\Leftarrow$) By the Lebesgue differentiation theorem, the limit of $m_I\omega$ as a sequence of intervals $I \in \mathcal{D}(J)$ shrinks to a point $x \in J$, is $\omega(x)$ for almost every x. $\square$

Let $\widetilde{I}$ be the parent of I. Denote by s_I the proportion of the mass of $\widetilde{I}$ that is carried by I. Then

$$2s_I = \frac{m_I\omega}{m_{\widetilde{I}}\omega} = 1 + b_{\widetilde{I}}h_{\widetilde{I}}(x), \qquad \forall\, x \in I. \tag{2.4}$$

2.2. **The λ-operation.** The nonlinear operation that sends the weight ω given by $\omega = \prod_{I\in\mathcal{D}}(1+b_Ih_I)$ into the weight ω_λ,

$$\omega \mapsto \omega_\lambda = \prod_{I\in\mathcal{D}}(1+\lambda b_Ih_I), \tag{2.5}$$

is called the λ-*operation*.

Remark 2.3. Dyadic doubling and $\mathbf{A}_\infty^d$ weights are preserved under the λ-operation for $-1 \leq \lambda \leq 1$. This is a consequence of Theorem 2.1, and the observation that since $|\lambda b_I| \leq |b_I|$, $b \in \mathbf{BMO}^d$ implies $\lambda b \in \mathbf{BMO}^d$; see [P2].

To understand how the λ-operation affects $\mathbf{RH}_p^d$ and $\mathbf{A}_p^d$ weights, let us first consider how it affects quotients of mean values of ω over consecutive nested intervals. Accordingly define

$$s_I(\lambda) = \frac{m_I\omega_\lambda}{2\, m_{\widetilde{I}}\omega_\lambda}. \tag{2.6}$$

The λ-operation is a nonlinear operation on the weight, but it is linear at the level of the s_I's:

Lemma 2.4. $s_I(\lambda) = \frac{1}{2} + \lambda(s_I - \frac{1}{2})$.

Proof: This is an immediate consequence of (2.4). $\square$

Lemma 2.5. *Let* $a_1, \ldots, a_n$ *be positive numbers. Let* $a_j(\lambda) = 1 + \lambda(a_j - 1)$, *and* $A_n(\lambda) = \prod_{j=1}^n a_j(\lambda)$. *Then*

$$A_n(\lambda) \geq \min\{1, A_n(1)\}. \tag{2.7}$$

Proof: On expanding the product we get

$$A_n(\lambda) = \prod_{k=1}^n \big((1-\lambda)+\lambda a_k\big) = \sum_{i=0}^n (1-\lambda)^{n-i}\lambda^i \sum_{k_1\neq\ldots\neq k_i}\prod_{j=1}^i a_{k_j}, \tag{2.8}$$

where the index notation $k_1 \neq \ldots \neq k_i$ means that the k_j's are pairwise different, and $1 \leq k_j \leq n$. Now, using repeatedly the fact that for positive numbers the arithmetic mean is at least the geometric mean, we find that

$$\sum_{k_1\neq\ldots\neq k_i}\prod_{j=1}^i a_{k_j} \geq N_{n,i}\left(\prod_{k_1\neq\ldots\neq k_i}\prod_{j=1}^i a_{k_j}\right)^{1/N_{n,i}}, \tag{2.9}$$

where $N_{n,i}$ is the number of ways one can choose i numbers from a collection of n numbers, that is, $N_{n,i} = \frac{n!}{i!(n-i)!}$. Observe that in the double product in (2.9), each a_j occurs as many times as possible ways one can choose $i-1$ numbers from a collection of $n-1$ numbers, that is $N_{n-1,i-1}$. Since $N_{n-1,i-1}/N_{n,i} = i/n$, the right hand side of (2.9) is equal to:

$$N_{n,i}\Big(\prod_{j=1}^{n} a_j\Big)^{i/n} \;=\; N_{n,i}\left(A_n(1)\right)^{i/n}. \tag{2.10}$$

We conclude that

$$A_n(\lambda) \;\geq\; \sum_{i=0}^{n}(1-\lambda)^{n-i}\lambda^i N_{n,i}\left(A_n(1)\right)^{i/n} \;=\; \Big(\lambda\big(A_n(1)\big)^{1/n} + (1-\lambda)\Big)^n.$$

Fix $A \geq 0$. The function $f(\lambda) = \big(\lambda A + (1-\lambda)\big)^n$ is monotonic for $\lambda \in [0,1]$, hence it is bounded below by $\min\{f(0), f(1)\}$. Set $A = \big(A_n(1)\big)^{1/n}$; then $f(0) = 1$, $f(1) = A_n(1)$. This completes the proof of the lemma. $\square$

3. The theorem for $\mathbf{A}_p^d$ weights

Theorem 3.1. *Take any p and λ such that $1 \leq p \leq \infty$ and $0 \leq \lambda \leq 1$. If the weight ω is in $\mathbf{A}_p^d$, then the weight ω_λ is also in $\mathbf{A}_p^d$.*

Proof: By Remark 2.3, ω_λ is a dyadic doubling weight, and $w \in \mathbf{A}_\infty^d$ implies $\omega_\lambda \in \mathbf{A}_\infty^d$.

For dyadic intervals $I \subset J$, the ratios $\frac{m_I\omega}{m_J\omega}$ and $\frac{m_I\omega_\lambda}{m_J\omega_\lambda}$ can be explicitly computed as products of s_K's and $s_K(\lambda)$'s respectively:

$$\frac{m_I\omega}{m_J\omega} = \prod_{I\subseteq K\subset J} 2s_K, \qquad \text{and} \qquad \frac{m_I\omega_\lambda}{m_J\omega_\lambda} = \prod_{I\subseteq K\subset J} 2s_K(\lambda), \tag{3.1}$$

where $K \in \mathcal{D}(J)$, $2s_K(\lambda) = 1 + \lambda(2s_K - 1)$, and $2s_K = m_K\omega/m_{\widetilde{K}}\omega$.

These are the kind of products considered in Lemma 2.5. Since ω is dyadic doubling, (2.4) shows that $\varepsilon \leq 2s_K \leq 2-\varepsilon$. In particular $a_K = 2s_K > 0$. We can apply Lemma 2.5 to each pair (I, J) of dyadic intervals, $I \subset J$, where $n = \log_2(|J|/|I|)$, $A_n(\lambda) = m_I\omega_\lambda/m_J\omega_\lambda$, and $A_n(1) = m_I\omega/m_J\omega$, to conclude that

$$\left(\frac{m_I\omega_\lambda}{m_J\omega_\lambda}\right)^r \leq \max\left\{1, \left(\frac{m_I\omega}{m_J\omega}\right)^r\right\}, \tag{3.2}$$

for all negative r.

Case $1 < p < \infty$: By Theorem 2.2(a), it suffices to find a constant C such that

$$\frac{1}{|J|}\sum_{I\in\mathcal{D}(J)}\left(\frac{m_I\omega_\lambda}{m_J\omega_\lambda}\right)^{\frac{-1}{p-1}} b_{I,\lambda}^2 \leq C \qquad \forall\, J \in \mathcal{D}, \tag{3.3}$$

where $b_{I,\lambda} = \langle \lambda b, h_I\rangle = \lambda b_I$.

We use (3.2), with $r = -1/(p-1)$, to estimate the left hand side of (3.3). Using the observation that $b_{I,\lambda}^2 \leq b_I^2$ for $0 < \lambda < 1$, we get

$$\begin{aligned}\frac{1}{|J|}\sum_{I\in\mathcal{D}(J)}\left(\frac{m_I\omega_\lambda}{m_J\omega_\lambda}\right)^{\frac{-1}{p-1}} b_{I,\lambda}^2 \;&\leq\; \frac{1}{|J|}\sum_{I\in\mathcal{D}(J)}\max\left\{1, \left(\frac{m_I\omega}{m_J\omega}\right)^{\frac{-1}{p-1}}\right\} b_I^2 \\ &\leq\; \frac{1}{|J|}\sum_{I\in\mathcal{D}(J)} b_I^2 + \frac{1}{|J|}\sum_{I\in\mathcal{D}(J)}\left(\frac{m_I\omega}{m_J\omega}\right)^{\frac{-1}{p-1}} b_I^2 \leq C;\end{aligned}$$

where C is a constant independent of the interval J. The last inequality holds since $\omega \in \mathbf{A}_p^d \subset \mathbf{A}_\infty^d$, by Theorems 2.1 and 2.2(a).

Case $p = 1$**:** By Theorem 2.2(b), it suffices to find a constant C such that

$$\left(\frac{m_I \omega_\lambda}{m_J \omega_\lambda}\right)^{-1} \leq C, \quad \forall\, I \in \mathcal{D}(J), \quad \forall\, J \in \mathcal{D}. \tag{3.4}$$

We use (3.2) again, with $r = -1$, to conclude that the left hand side of (3.4) is bounded by $\max\left\{1, \left(\frac{m_I w}{m_J w}\right)^{-1}\right\}$, which in turn is bounded because $\omega \in \mathbf{A}_1^d$. □

4. Examples in $\mathbf{RH}_p^d$

Theorem 4.1. *For each $p > 1$, there exist a dyadic doubling weight ω in $\mathbf{RH}_p^d$ and a number $\lambda \in (0,1)$ such that ω_λ is not in $\mathbf{RH}_p^d$.*

Proof: Our examples are of the following form. Let $I_i = [2^{-i}, 2^{-i+1}]$ for $i \geq 1$, and $J_i = [0, 2^{-i}]$ for $i \geq 0$. Fix numbers $s_i \in (0,1)$ for $i \geq 1$. For $x \in [0,1]$ let

$$\omega(x) = \sum_{i=1}^{\infty} c_i\, \chi_{I_i}(x), \quad \text{where} \quad c_i = 2^i\, s_1 \ldots s_{i-1}(1 - s_i). \tag{4.1}$$

One can think of ω as the weight of mass one obtained by assigning the fractions s_i and $1 - s_i$ of the mass of J_{i-1} to the left half J_i and the right half I_i respectively of J_{i-1}, for each $i \geq 1$. In the notation of Section 2,

$$s_i = s_{J_i} = \frac{m_{J_i} \omega}{2\, m_{J_{i-1}} \omega}, \quad \text{and} \quad m_{J_i}\omega = 2^i\, s_1 \ldots s_{i-1} s_i. \tag{4.2}$$

We further assume that the sequence $\{s_i\}$ is *n-periodic*: there is a smallest positive integer n such that $s_{i+n} = s_i$ for all $i \geq 1$. Such a ω is a dyadic doubling weight. For n-periodic weights of this form,

$$c_{kn+j} = (2^n\, s_1 \ldots s_n)^k\, c_j, \qquad \forall\, k \geq 0, j \geq 1. \tag{4.3}$$

Claim: *The $\mathbf{RH}_p^d$ condition $\frac{1}{|I|}\int_I \omega^p \leq C\left(\frac{1}{|I|}\int_I \omega\right)^p$, for all $I \in \mathcal{D}$, for an n-periodic weight of the form (4.1) reduces to the condition*

$$2^n s_1 \ldots s_n < 2^{n/p}. \tag{4.4}$$

To see this, first note that ω is constant on each dyadic interval I whose left endpoint is not 0, so the reverse Hölder p condition holds with $C = 1$ on these intervals. The only other dyadic intervals are $J_l = [0, 2^{-l}]$, $l \geq 0$.

Write $l = qn + m$ where q and m are non-negative integers and $0 \le m \le n-1$. We use (4.3) to compute the mean of ω^p on $J_l = \bigcup_{i=l+1}^{\infty} I_i$:

$$\begin{aligned}
\frac{1}{|J_l|}\int_{J_l}\omega^p &= 2^l \sum_{i=l+1}^{\infty}\int_{I_i}\omega^p = 2^l \sum_{i=qn+m+1}^{\infty} c_i^p\, 2^{-i} \\
&= 2^l \sum_{k=q}^{\infty}\sum_{j=1}^{n} (c_{kn+j+m})^p\, 2^{-(kn+j+m)} \\
&= 2^l \sum_{k=q}^{\infty}\sum_{j=1}^{n} (2^n s_1 \dots s_n)^{kp}\, (c_{j+m})^p\, 2^{-(kn+j+m)} \\
&= 2^l\, d_{p,m} \sum_{k=q}^{\infty} \left[(2^n\, s_1 \dots s_n)^p\, 2^{-n}\right]^k, \qquad (4.5)
\end{aligned}$$

where
$$d_{p,m} = \sum_{j=1}^{n} (c_{j+m})^p\, 2^{-(j+m)}.$$

The series (4.5) converges if and only if $2^n\, s_1 \dots s_n < 2^{n/p}$; this is condition (4.4).

The mean of ω on J_l is

$$\frac{1}{|J_l|}\int_{J_l}\omega = 2^l\, d_{1,m} \sum_{k=q}^{\infty} (s_1 \dots s_n)^k\,; \qquad (4.6)$$

this series converges since each s_i is less than one.

If $2^n\, s_1 \dots s_n < 2^{n/p}$, then on summing the series in (4.5) and (4.6) we find

$$\frac{\frac{1}{|J_l|}\int_{J_l}\omega^p}{\left(\frac{1}{|J_l|}\int_{J_l}\omega\right)^p} = 2^{l(1-p)}\, \frac{d_{p,m}}{(d_{1,m})^p}\, 2^{n(p-1)q}\, \frac{[1-(s_1 \dots s_n)]^p}{1-(2^n s_1 \dots s_n)^p\, 2^{-n}}, \qquad (4.7)$$

for all $l = qn + m \ge 0$ with $q \ge 0$ and $0 \le m \le n-1$. This expression depends on p, n, and m but is independent of q, since $d_{p,m}$ and $d_{1,m}$ are independent of q and the exponent of 2 in the right hand side of (4.7) reduces to $m(1-p)$. Taking the maximum over $m \in \{0, \dots, n-1\}$, we see that there is a uniform upper bound, depending only on p and n, for $\frac{1}{|J_l|}\int_{J_l}\omega^p / (\frac{1}{|J_l|}\int_{J_l}\omega)^p$, for all $l \ge 0$. Therefore $\omega \in \mathbf{RH}_p^d$, which establishes the claim.

Let $f(x_1, \dots, x_n) = 2^n\, x_1 \dots x_n$. Then by the claim above, the n-periodic weight ω associated to the point $(s_1, \dots, s_n)$ in the open unit n-cube $(0,1)^n$ is in $\mathbf{RH}_p^d$ if and only if $f(s_1, \dots, s_n) < 2^{n/p}$.

By Lemma 2.4, the λ-operation takes the weight ω with sequence $\{s_i\}$ to a new weight ω_λ with sequence $\{s_i(\lambda)\}$, where

$$s_i(\lambda) = \frac{1}{2} + \lambda\left(s_i - \frac{1}{2}\right). \qquad (4.8)$$

In the unit n-cube, the λ-operation moves the point $P : (s_1, \dots, s_n)$ towards the centre $Q : (1/2, \dots, 1/2)$ of the cube along the line segment L joining P to Q. For $\lambda \in (0,1)$, the point $P_\lambda : \big(s_1(\lambda), \dots, s_n(\lambda)\big)$ associated to ω_λ lies on L between P and Q.

We wish to find a weight ω and a number $\lambda \in (0,1)$ such that $\omega \in \mathbf{RH}_p^d$ but $\omega_\lambda \notin \mathbf{RH}_p^d$. To do this, we will exhibit a line L through Q, and points P and P_λ on L, such that: (i) P and P_λ are inside the open unit cube $(0,1)^n$, and P_λ is between P and Q; (ii) the value of f at P is strictly less than $2^{n/p}$; and (iii) the value of f at P_λ is at least $2^{n/p}$. Then: $(i)'$ there is a unique $\lambda \in (0,1)$ such that the λ-operation takes the weight ω associated to P to the weight ω_λ associated to P_λ; $(ii)'$ the weight ω associated to P is in $\mathbf{RH}_p^d$; and $(iii)'$ the weight ω_λ associated to P_λ is not in $\mathbf{RH}_p^d$.

Fix $a \in (0, 1/2)$. Let L be the line through $(a, 1, \dots, 1)$ and Q; it has equation $\left(a + \left(\frac{1}{2} - a\right) t, 1 - \frac{t}{2}, \dots, 1 - \frac{t}{2}\right)$, and values of $t \in [0,2]$ correspond to points in the closed unit cube $[0,1]^n$. Let $g(t)$ be the restriction of f to L, for $t \in [0,2]$:

$$g(t) = \big(2a + (1-2a)t\big)(2-t)^{n-1}. \tag{4.9}$$

Then $g'(t) = (2-t)^{n-2}\left[-n\,(1-2a)\,t + 2\,(1-(n+1)\,a)\right]$.

If $n > 2$, g has two critical points; if $n = 2$, g has a single critical point. Since $g(2) = f(1-2a, 0, \dots, 0) = 0$, and f is positive in the open unit cube, a critical point of g at $t = 2$ is a minimum. The critical point which occurs at

$$t_m = \frac{2\,(1-(n+1)\,a)}{n(1-2a)} \tag{4.10}$$

is a maximum for g. Denote this point by P_m. The maximum value is

$$g(t_m) = \frac{2^n\,(n-1)^{n-1}\,(1-a)^n}{n^n\,(1-2a)^{n-1}}. \tag{4.11}$$

This maximum is taken inside the open unit cube if and only if $t_m > 0$; in other words, if and only if $a < (n+1)^{-1}$. In this case, P_m lies between $(a, 1, \dots, 1)$ and the centre Q of the cube.

The maximum value of g is a continuous, increasing function of a, for $a \in [0, (n+1)^{-1}]$. We denote this function by $h(a) = g(t_m)$ (see (4.11) above). Then $h'(a) > 0$ for $a \in [0, (n+1)^{-1}]$. Also, $h(0) = 2^n\,(n-1)^{n-1} n^{-n}$, and $h\big(\frac{1}{n+1}\big) = \frac{2^n}{n+1}$. Therefore, if $p > 1$ is large enough that

$$2^{n/p} < \frac{2^n}{n+1}, \tag{4.12}$$

then we can choose $a_p \in (0, (n+1)^{-1})$ so that $h(a_p) \geq 2^{n/p}$ and so that the maximum value $h(a_p)$ of f along the line segment from $(a_p, 1, \dots, 1)$ to Q is taken on at a point inside the open unit cube.

Notice that if p is too large, we may not be able to find a line segment on which the maximum value $h(a_p)$ of f is actually equal to $2^{n/p}$, since the smallest $h(a_p)$ is $h(0)$ which is strictly larger than one (if $n > 2$).

Define P_λ as follows: if $h(0) < 2^{n/p} < \frac{2^n}{n+1}$, then choose the unique $a_p \in (0, (n+1)^{-1})$ such that $h(a_p) = 2^{n/p}$, and let $P_\lambda = P_m$ for that a_p. If $2^{n/p} \leq h(0)$, choose a_p small enough that $f(a_p, 1, \dots, 1) = 2^n\,a_p < 2^{n/p}$, and let $P_\lambda = P_m$ for that a_p.

In both cases, $f(a_p, 1, \dots, 1) < 2^{n/p}$ while the value of f at P_m is at least $2^{n/p}$. Since g is increasing for $0 < t < t_m$, we may choose a point P between $(a_p, 1, \dots, 1)$ and P_λ such that the value of f at P is strictly less than $2^{n/p}$. Let λ be the unique number in $(0,1)$ such that the λ-operation takes this point P to P_λ.

The weights ω and ω_λ associated to P and P_λ respectively, together with the number λ, furnish our example: ω is in $\mathbf{RH}_p^d$ but ω_λ is not in $\mathbf{RH}_p^d$.

The critical value of p, from (4.12), is

$$p_c = \frac{n \log 2}{n \log 2 - \log(n+1)}. \tag{4.13}$$

Examples for all $p > p_c$ can be found among the n-periodic weights of the form described above. Letting n tend to infinity, we see that p_c tends to 1, and so we have found examples for all $p > 1$. $\square$

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