On the Nonexistence of Certain Divergence-free Multi-wavelets

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Abstract. We show that there are no biorthogonal pairs of divergence-free multi-wavelet families on $\mathbb{R}^n$, having any regularity, such that both biorthogonal families have compactly supported, divergence-free generators. This main result generalizes Lemarié's bivariate result. In particular, our method is based on vector-valued multiresolution analyses.

1. Introduction

We will demonstrate the nonexistence of biorthogonal (multi)-wavelets having some regularity, such that both biorthogonal families have compactly supported, divergence-free generators. This gives a negative answer to a question posed to us concerning the possible use of multiwavelets to circumvent the nonexistence result of Lemarié [3]. Actually, this fact does not impart severe obstacles to the application of wavelet-Galerkin methods for Navier-Stokes systems [5]. Nevertheless, it clarifies the sort of limitations that the wavelets must have, cf., [4].

The method is based on Lemarié's bivariate result [3] which relies on a dimensionality trick: divergence-free vector fields in two dimensions have the form $(\partial f/\partial x_2, -\partial f/\partial x_1)$ where $f : \mathbb{R}^2 \to \mathbb{R}$ is a scalar-valued function. This trick goes a long way toward making the analysis essentially scalar-valued, and is not available in higher dimensions. Lemarié's method is based, in turn, on his characterization of projectors onto shift invariant spaces [2] together with a key result of DeVore, DeBoor and Ron [1]. Since Lemarié's trick is not available in dimension three or higher, to extend his technique requires extending results in [2] and [1] to a subspace $\mathcal{H}^2 = \mathcal{H}^2(\mathbb{R}^n, \mathbb{R}^n)$ of $L^2(\mathbb{R}^n, \mathbb{R}^n)$ consisting of distributionally divergence-free vector fields. Here $\mathcal{H}$ stands for *Hardy*. The required extensions are done here. One of the main factors that allows the analysis to go through is the nontrivial fact that $\mathcal{H}^2$ is preserved under scalar-valued Fourier multipliers of L^2. We restrict our wavelet analysis to the case of two-scale dilations: more general dilations that map integers to integers can be considered as in [2].

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2. The Main Theorem

First, we need some conventions. Unless explicitly stated, all functions of x will map $\mathbb{R}^n \to \mathbb{R}^n$ and their values will be regarded as column vectors, their transposed values as row vectors. Then $f^\top g$ will be scalar-valued; in particular, the L^2 inner product is $\langle f, g\rangle = \int f^\top g$. But $fg^\top$ will take values in the $n \times n$ matrices $\mathcal{M}_n(\mathbb{R})$. The Fourier transform is defined on integrable functions by the formula $\widehat{f}(\xi) = \int_{\mathbb{R}} e^{-2\pi i x\cdot\xi} f(x)dx \in \mathbb{C}^n$. The integration in this case is done componentwise on f. However, when we represent an operator K by integration against a kernel $k(x, y)$, the kernel will typically have values in $\mathcal{M}_n(\mathbb{R})$.

Divergence-free wavelet families coming from two-scale dilations will be indexed by sets of the form

$$\Lambda = \{\lambda = (\varepsilon, \alpha, i, Q) : \varepsilon \in \{0,1\}^n \setminus \{0, 0\}, i = 1, ..., n-1, Q \in \mathcal{Q}, \alpha \in \mathcal{A}\}$$

where $\mathcal{Q}$ denotes the collection of dyadic cubes, and $\mathcal{A}$ is some finite index that is used to indicate that the wavelets are allowed to be 'multi'-wavelets. Later, $Q(\lambda)$ will denote the dyadic cube belonging to the index λ. By a biorthogonal pair of wavelet bases for the space $\mathcal{H}^2(\mathbb{R}^n, \mathbb{R}^n)$ we mean a pair $\{\psi_\lambda, \psi^*_{\lambda'}\}_{\lambda,\lambda'\in\Lambda}$ such that any $f \in \mathcal{H}^2(\mathbb{R}^n, \mathbb{R}^n)$ possesses an expansion of the form

$$f = \sum_{\lambda\in\Lambda} \langle f, \psi^*_\lambda\rangle\, \psi_\lambda;\ \ \|f\|_2^2 \sim \sum_{\lambda\in\Lambda} |\langle f, \psi^*_\lambda\rangle|^2 .$$

The square-integrable vector fields ψ_λ are divergence-free in the sense of distributions modulo polynomials, but the ψ^*_λ need not be. In fact, the main result is nothing but the statement that under certain niceness conditions they *cannot* be:

Theorem 1. *Under the hypotheses above, the biorthogonal families $\{\psi_\lambda, \psi^*_{\lambda'}\}$ cannot both be divergence-free, compactly supported, and Hölder continuous of some positive order.*

The result does not pose obstacles to wavelet-Galerkin methods because the *synthesizing* wavelets have the differential property built in. The only property required of the *analyzing* wavelets is that they allow wavelet coefficients to be computed rapidly.

Here is the idea of the proof: Gradients of harmonic, scalar-valued functions h are locally square-integrable. In particular, if the ψ^*_λ have compact support then the coefficients $\langle \nabla h, \psi^*_\lambda\rangle$ are well-defined and, when $\nabla\cdot\psi^*_\lambda = 0$, they vanish via integration by parts. As it turns out, this allows one to identify the restriction of a certain cutoff of ∇h with the restriction of its 'V_0' multiresolution projection to a neighborhood of the origin. If the ψ_λ have compact support then the space V_0 turns out to be locally finite dimensional. Most of the work lies in verifying this. In contrast, harmonic functions form a locally infinite dimensional space. This contradicts the local equality between ∇h and its V_0-projection. Regularity and compact support of the ψ_λ^* is required to apply integration by parts; it is required of the ψ_λ to conclude that V_0 is locally finite dimensional.

3. Shift invariant subspaces of $\mathcal{H}^2(\mathbb{R}^n, \mathbb{R}^n)$ and ω-localized projections

Much of this section amounts to lifting known facts about $L^2(\mathbb{R}^n)$ to the case of $\mathcal{H}^2(\mathbb{R}^n, \mathbb{R}^n)$. First, a closed subspace V_0 of $\mathcal{H}^2(\mathbb{R}^n, \mathbb{R}^n)$ is said to be *shift invariant* if, whenever $f \in V_0$ and $k \in \mathbb{Z}^n$, one has $f(x-k) \in V_0$. In what follows V_0 will always be shift invariant. The operator P_0 that projects onto V_0 is shift invariant if for all $f \in \mathcal{H}^2(\mathbb{R}^n, \mathbb{R}^n)$ one has $(P_0 f)(x-k) = P_0(f(\cdot - k))(x)$. When is V_0 spanned by the translates $\phi_\alpha(x-k)$ of a finite family $\{\phi_\alpha\}_{\alpha \in A}$? Such a space is said to be a *finite shift invariant* (FSI) space. More particularly, when does $\{\phi_\alpha(x-k)\}_{\alpha \in A, k \in \mathbb{Z}^n}$ form a Riesz basis for V_0?

The answer can be phrased in terms of regularity of the kernel of P_0. This was formulated for $L^2(\mathbb{R}^n)$ by Lemarié. We say that P_0 has an *ω-localized kernel for $\mathcal{H}^2$* provided its action is given by integration against a locally integrable kernel function $p(x,y) \in \mathcal{M}_n$ such that $\nabla_x^\top \cdot p(x,y) = 0$ in the sense that

$$\langle P_0 f, g\rangle = \int\int \overline{g}^\top(x) p(x,y) f(y)\, dy\, dx; \quad \langle P_0 f, \nabla_x h\rangle = 0, \quad h \in C_c^\infty(\mathbb{R}^n)$$

where

$$\begin{aligned} \int_{x\in[0,1)^n}\int_{y\in\mathbb{R}^n} \omega(x-y)|p(x,y)|^2\, dx\, dy &< \infty \\ \int_{y\in[0,1)^n}\int_{x\in\mathbb{R}^n} \omega(x-y)|p(x,y)|^2\, dx\, dy &< \infty. \end{aligned}$$

Here, $|p(x,y)| = [\sum_{i,j=1}^n |p_{ij}(x,y)|^2]^{1/2}$. We will need to assume that ω is a symmetric *Beurling weight*. This means that ω is a nonegative valued function that is strictly bounded below, has growth of at most polynomial order, $\omega(x) = \omega(-x)$, $\omega(x+y) \le \omega(x)\omega(y)$ and $\omega^{-1} \in L^1(\mathbb{R}^n)$ and $\omega^{-1} * \omega^{-1} \le C\omega^{-1}$. For a review of the properties of Beurling weights we refer to Annexe A in [2]. We have:

Theorem 2. *If P_0 has an ω-localized kernel for $\mathcal{H}^2$ then V_0 has a Riesz basis of the form $\{\phi_\alpha(x-k)\}_{\alpha\in\mathcal{A}, k\in\mathbb{Z}^n}$. The cardinality $A = |\mathcal{A}|$ does not depend on the choice of the basis. In fact,*

$$A = \operatorname{tr}\int_{[0,1)^n}\int_{[0,1)^n} \widetilde{p}(x,y)\widetilde{p}(y,x)\; dy\, dx, \qquad \text{where}$$

$$\widetilde{p}(x,y) \;=\; \sum_{k\in\mathbb{Z}^n} p(x,y-k).$$

Note that A is well defined because $\widetilde{p}$ is periodic and locally square-integrable. The kernel of P_0 can be defined *a-posteriori* by the basis expansion:

$$p(x,y)=\sum_k\sum_\alpha \phi_\alpha(x-k)\phi_\alpha(y-k)^\top$$

which is $\mathcal{M}_n$-valued when ϕ is $\mathbb{R}^n$-valued. However, we must construct the basis given the ω-localized kernel first.

The appearance of Beurling weights in this theory might look a bit peculiar, so we will briefly remark on their role here. The basis functions ϕ_α will be assembled from certain functions in the weighted space L^2_ω which is a subspace of the Beurling algebra. The crucial step in the assembly is a local inversion at the Fourier level, which is made possible because elements of the Beurling algebra have absolutely convergent Fourier transforms. In the case of Theorem 1, the Beurling weight that arises is just $(1+|x|)^{n+\rho}$ where ρ depends on the Hölder regularity.

Before proceeding, it makes sense to introduce here the analogue of an additional key idea due to DeVore, de Boor, and Ron [1] which addresses the remaining crucial ingredient for Theorem 1: the role of compact support. In [1] a FSI space is called *local* if it has compactly supported generators $\{\phi_\alpha\}$. Local FSI spaces are locally finite-dimensional. That is, given any compact set there is a finite family of functions in the space whose restrictions to that set span the restrictions of all elements of the space to that set. The following corresponds to Corollary 3.36 of [1]:

Theorem 3. *Let V_0 be a finite shift invariant subspace of $\mathcal{H}^2$, and suppose that the compactly supported elements of V_0 are dense in V_0. Then V_0 is local.*

In the case of a multiresolution analysis and, in particular, in the setup of Theorem 1, V_0 can be identified with the closed linear span of the wavelets living on long scales. Then V_0 is local when the wavelets have compact support.

3.1. The correlation function. Given $f,g\in\mathcal{H}^2(\mathbb{R}^n,\mathbb{R}^n)$, we define their *correlation function*

$$C(f,g)(\xi)=\sum_{k\in\mathbb{Z}^n}\widehat{f}(\xi+k)^\top\overline{\widehat{g}}(\xi+k)$$

which is a periodic, $\mathbb{C}$-valued function. We call $C(f,f)=C(f)$ the *autocorrelation function* of f. When f,g are square-integrable their autocorrelation functions are integrable over the unit cube $[0,1)^n$. Hence, by Cauchy-Schwarz, $C(f,g)\in L^1([0,1)^n)$. The next three lemmas are just $\mathbb{R}^n$-valued versions of Lemmas 1-3 in section I.1 of [2].

Lemma 4. *The family $\{f(\cdot - k)\}$ is a Bessel family if and only if $C(f) \in L^\infty(\mathbb{T}^n)$.*

Proof. The point is that $C(f)$ can be regarded as a multiplier of $L^2(\mathbb{T}^n)$. Thus, if $\{\alpha_k\} \in l^2(\mathbb{Z}^n)$, Plancherel's theorem gives

$$\left\|\sum \alpha_k f(\cdot - k)\right\|_2^2 = \int_{[0,1)^n} \left|\sum \alpha_k e^{-2\pi i k\cdot\xi}\right|^2 C(f)(\xi)d\xi \le C\left\|\{\alpha_k\}\right\|_{l^2}^2$$

with C independent of $\{\alpha_k\}$ if and only if $C(f) \in L^\infty(\mathbb{T}^n)$. ∎

Recall that a family of functions $\{f_\kappa\}_{\kappa\in K}$ is a Riesz family if whenever $\{\alpha_\kappa\} \in l^2(K)$ one has $\|\{\alpha_\kappa\}\|_{l^2} \le C\left\|\sum \alpha_\kappa f_\kappa\right\|_2$. When $K = \mathcal{A}\times\mathbb{Z}^n$ for a fixed, finite set $\mathcal{A}$, $\{f_\kappa\}_{\kappa\in K} = \{f_\alpha(\cdot - k)\}_{\alpha\in\mathcal{A},k\in\mathbb{Z}^n} \equiv \{f_\alpha(\cdot - k)\}$. Then one can form the *correlation matrix* $M(\xi) = \{C(f_\alpha, f_{\alpha'})\}_{\alpha,\alpha'} \in \mathcal{M}_A$ where $A = |\mathcal{A}|$. We shall write $M(\xi) = M_{f_\alpha}(\xi)$ when we wish to distinguish the correlations of a specific family $\{f_\alpha\}_{\alpha\in\mathcal{A}}$ from those of another family. We want to address the issue of when $\{f_\alpha(\cdot - k)\}$ is a Riesz basis (i.e., Bessel family and Riesz family) for V_0. For any such basis there exists a basis $\{f^*_\kappa\}_{\kappa\in K}$ for $V_0^* = (\ker P_0)^\perp$ that is biorthogonal to $\{f_\alpha(\cdot - k)\}$ in the sense that $\left\langle f^*_{\alpha',k'}, f_\alpha(\cdot - k)\right\rangle = \delta_{\alpha\alpha'}\delta_{kk'}$. Actually, the uniqueness of this basis implies that $f^*_{\alpha,k}(x) \equiv f^*_\alpha(x-k)$. We shall determine the functions $\{f^*_\alpha\}$ explicitly in terms of $\{f_\alpha\}$ but first we shall use their existence to formulate a useful condition for determining whether $\{f_\alpha(\cdot - k)\}$ is a Riesz basis:

Lemma 5. *The family $\{f_\alpha(\cdot - k)\}$ is a Riesz basis for its span if and only if $M(\xi) \in \mathcal{M}_A(L^\infty)$ and $1/\det M(\xi) \in L^\infty$.*

Proof. First assume that $\{f_\alpha(\cdot - k)\}$ is a Riesz basis for its span and let $\{f^*_\alpha(\cdot - k)\}$ be the dual basis of $\{f_\alpha(\cdot - k)\}$. Since

$$f^*_\alpha = \sum_k \sum_{\alpha'} \left\langle f^*_\alpha, f^*_{\alpha'}(\cdot - k)\right\rangle f_{\alpha'}(\cdot - k),$$

one has:

$$\widehat{f^*_\alpha}(\xi) = \sum_k \sum_{\alpha'} \left\langle f^*_\alpha, f^*_{\alpha'}(\cdot - k)\right\rangle e^{-2\pi i k\cdot\xi}\widehat{f_{\alpha'}}(\xi).$$

But, by Parseval's formula, $\langle f^*_\alpha, f^*_{\alpha'}(\cdot - k)\rangle$ is the k-th Fourier coefficient of $C(f^*_\alpha, f^*_{\alpha'})$, whence

$$\widehat{f^*_\alpha}(\xi) = \sum_{\alpha'} C(f^*_\alpha, f^*_{\alpha'})(\xi)\widehat{f_{\alpha'}}(\xi).$$

By a similar computation one concludes that

$$C(f^*_\alpha, f_{\alpha'})(\xi) = \sum_{\alpha''} C(f^*_\alpha, f^*_{\alpha''})(\xi)C(f_{\alpha''}, f_{\alpha'})(\xi).$$

On the other hand, biorthogonality implies that

$$C(f^*_\alpha, f_{\alpha'})(\xi) = \sum_k \langle f^*_\alpha, f_{\alpha'}(\cdot - k)\rangle\, e^{-2\pi i k\cdot\xi} = \delta_{\alpha\alpha'}.$$

Therefore, $(M_{f_\alpha}(\xi))^{-1} = M_{f^*_\alpha}(\xi)$ where the latter matrix is the correlation matrix for the dual generators. Now the $\{f^*_\alpha(x-k)\}$ form a Bessel family because they are also a Riesz basis for their span. By the previous lemma, together with Cauchy-Schwarz, it follows that both $M_{f_\alpha}(\xi) \in \mathcal{M}_A(L^\infty)$ and $M_{f^*_\alpha}(\xi) \in \mathcal{M}_A(L^\infty)$. In particular,

$$1 = (\det M_{f_\alpha}(\xi)) \left(\det M_{f^*_\alpha}(\xi)\right) \le \det M_{f_\alpha}(\xi) \left\|\det M_{f^*_\alpha}(\xi)\right\|_\infty$$

so that

$$\det M_{f_\alpha}(\xi) \ge \frac{1}{\left\|\det M_{f^*_\alpha}(\xi)\right\|_\infty}.$$

As the right hand side is independent of ξ, the "only if" follows.

To prove the "if" part, assume that $M(\xi) \in \mathcal{M}_A(L^\infty)$ and that $\det M(\xi)$ is essentially bounded below. Since $(f,g) \mapsto C(f,g)(\xi)$ is Hermitian, it follows that $M(\xi)$ is a positive definite Hermitian matrix almost everywhere. One concludes that there exists a function $\gamma(\xi)$ at least as large as the *smallest* eigenvalue of $M(\xi)$ such that, for any vector $\lambda = \{\lambda_\alpha\}_{\alpha\in\mathcal{A}}$, one has $\lambda^\top M(\xi)\overline{\lambda} \ge \gamma(\xi)\sum_\alpha |\lambda_\alpha|^2$. On the other hand, if $\Lambda(\xi)$ is the *largest* eigenvalue of $M(\xi)$ then

$$\gamma(\xi) \ge \frac{\det M(\xi)}{\Lambda(\xi)^{A-1}}.$$

Since $\Lambda(\xi) \le \left[A\sum_\alpha C(f_\alpha)(\xi)\right]^{1/2}$, $\gamma(\xi)$ is minorized by some fixed constant $\gamma > 0$. Thus, setting $\nu_\alpha(\xi) = \sum_k \lambda_{k,\alpha} e^{-2\pi i k\cdot\xi}$ and $\nu(\xi) = \overrightarrow{\nu_\alpha}(\xi) \in \mathbb{C}^A$ and applying Plancherel yields

$$\begin{aligned}
\left\|\sum_k\sum_\alpha \lambda_{k,\alpha} f_\alpha(x-k)\right\|_2^2 &= \int_{[0,1)^n} \vec{\nu}^\top(\xi) M(\xi)\overline{\vec{\nu}}(\xi)\, d\xi \\
&\ge \gamma \int_{[0,1)^n} \sum_\alpha |\nu_\alpha(\xi)|^2\, d\xi \\
&= \gamma \sum_\alpha\sum_k |\lambda_{k,\alpha}|^2
\end{aligned}$$

which shows that the family $\{f_\alpha(x-k)\}$ forms a Riesz basis for its span. ■

In general, one can define the correlation matrix $M(\{f_\alpha\},\{g_\beta\})(\xi)$ between two different families of generators $\{f_\alpha\}_{\alpha\in\mathcal{A}}$ and $\{g_\beta\}_{\beta\in\mathcal{B}}$: once again the entries are the correlations between the various generators. We take this approach presently because it will give a precise description of V_0 as a complemented subspace.

We say that two subspaces V, W of $\mathcal{H}^2$ are in duality provided $\mathcal{H}^2 = V \oplus W^\perp$.

Lemma 6. *Let V and W be two closed subspaces of $\mathcal{H}^2$ such that V has a Riesz basis of the form $\{f_\alpha(x-k)\}$ and W has a Riesz basis of the form $\{g_\beta(x-k)\}$. Also let $\{\phi_\delta\}_{\delta\in\mathcal{D}} \subseteq V$ and $\{\psi_\tau\}_{\tau\in\mathcal{T}} \subseteq W$. Then:*

(a) $M(\{\phi_\delta\},\{\psi_\tau\}) = M(\{\phi_\delta\},\{f_\alpha^*\})M(\{f_\alpha\},\{g_\beta\})M(\{g_\beta^*\},\{\psi_\tau\})$ where $\{f_\alpha^*\},\{g_\beta^*\}$ denote the bases dual to $\{f_\alpha\},\{g_\beta\}$ in V,W respectively,

(b)$\{\phi_\delta(x-k)\}$ is a Riesz basis of V if and only if $|\mathcal{D}|=|\mathcal{A}|$ and $M(\{\phi_\delta\},\{f_\alpha^*\}) \in \mathcal{M}_A(L^\infty)$ with inverse in $\mathcal{M}_A(L^\infty)$,

(c) If $N(\xi) = [M(\{f_\alpha\},\{f_\alpha\})(\xi)]^{-1/2}$ and $\widehat{\phi_\alpha}(\xi) = \sum_{\alpha'} N_{\alpha,\alpha'}(\xi)\widehat{f_{\alpha'}}(\xi)$ then the $\{\phi_\alpha(x-k)\}$ forms an orthonormal basis of V.

(d) V,W are in duality if and only if $|\mathcal{A}|=|\mathcal{B}|$ and $M(\{f_\alpha\},\{g_\beta\}) \in \mathcal{M}_A(L^\infty)$. Moreover, the dual basis $\{\gamma_\alpha^*\}$ of $\{f_\alpha\}$ in W is calculated via $\widehat{\gamma_\alpha^*}(\xi) = \sum_\beta \Gamma_{\alpha,\beta}(\xi)\widehat{g_\beta}(\xi)$ where the matrix Γ satisfies $M(\{f_\alpha\},\{g_\beta\})^\top\overline{D} = I$.

Proof. The proof of (a) follows immediately from the fact that $\widehat{\phi_\eta} = \sum_\alpha C(\phi_\eta, f_\alpha^*)\widehat{f_\alpha}$ and a similar identity represents ψ_τ. If $\{\phi_\delta(x-k)\}$ forms a Riesz basis of V then $M_{\phi_\delta} = M(\{\phi_\delta\},\{\phi_\delta\})$ has maximal rank $|\mathcal{D}|$ almost everywhere. Because one can factorize $M_{f_\alpha} = M(\{f_\alpha\},\{\phi_\delta^*\})M_{\phi_\delta}M(\{\phi_\delta^*\},\{f_\alpha\})$ and one can factorize M_{ϕ_δ} in terms of $\{f_\alpha\}$ similarly, maximal rank of each implies $|\mathcal{D}|=|\mathcal{A}|=A$. Using factorization to compute determinants yields

$$\det M_{\phi_\delta} = \det M_{f_\alpha}\,|\det M(\{\phi_\delta\},\{f_\alpha^*\})|^2$$

By Lemma 5, $|\det M(\{\phi_\delta\},\{f_\alpha^*\})|$ is minorized a.e. by a fixed constant; in particular, $M(\{\phi_\delta\},\{f_\alpha^*\})$ is uniformly invertible. The converse is immediate from Lemma 5. This proves (b).

To verify (c) first we show that $\{\phi_\alpha(x-k)\}$ forms a Riesz basis of V. By (a) and biorthogonality of $\{f_\alpha\},\{f_\alpha^*\}$, $M(\{\phi_\alpha\},\{f_\alpha^*\})(\xi) = N(\xi)$, hence by multiplicativity of determinants,

$$\det M(\{\phi_\alpha\},\{f_\alpha^*\}) = \frac{1}{\sqrt{\det M(\{f_\alpha\},\{f_\alpha\})}}.$$

Next we wish to show that

$$N(\xi) = \frac{2}{\pi}\int_0^\infty (I+t^2M(\xi))^{-1}dt$$

has bounded coefficicients. Let $\lambda(\xi)$ and $\Lambda(\xi)$ denote the smallest and largest eigenvalues of $M(\xi)$. Then

$$(1+t^2\lambda(\xi))^A \le \det(I+t^2M(\xi)) \le (1+t^2\Lambda(\xi))^A$$

and, consequently,

$$\left\|(I+t^2M(\xi))^{-1}\right\|_{M_A(L^\infty)} \le \frac{C}{1+t^2},$$

hence $N(\xi) \in M_A(L^\infty)$. This proves that $\{\phi_\alpha(x-k)\}$ is a Riesz basis of V. To show that $\{\phi_\alpha(x-k)\}$ is an orthonormal family, it is enough to show that $M(\{\phi_\delta\},\{\phi_\delta\}) = I$. But, by (a),

$$M(\{\phi_\delta\},\{\phi_\delta\}) = N(\xi)M(\{f_\alpha\},\{f_\alpha\})N(\xi) = I.$$

This proves (c).

Finally, (d) is clear since V, W are in duality if and only if W has a Riesz basis $\{\gamma^*_\alpha(x-k)\}_\alpha$ such that $M(\{f_\alpha\},\{\gamma^*_\alpha\}) = I_A$. ■

3.2. Proof of Theorem 2. We assume here that P_0 has an ω-localized kernel $p(x,y)$. The estimate just below and the proof of Lemma 7 will rely, in turn, on properties of Beurling weights described in Lemma 12 of [2]. First, recall that $\widetilde{p}(x,y) = \sum_k p(x,y-k)$, is periodic in both x,y because of shift invariance: $p(x,y-k) = p(x+k,y)$. To verify local square-integrability, taking p in $\mathbb{C}^{n\times n}$, Cauchy-Schwarz gives

$$\begin{aligned}\int_{[0,1)^{2n}} \left|\sum_k p(x,y-k)\right|^2 \, dx\,dy &\leq \int_{[0,1)^{2n}} \sum_k |p(x,y-k)|^2\,\omega(x-y+k)\,dx\,dy \\ &\times \sup_{(x,y)\in[0,1)^{2n}} \left|\sum_k \frac{1}{\omega(x-y+k)}\right| \\ &= C\int_{[0,1)^n}\int_{\mathbb{R}^n} |p(x,y)|^2\,\omega(x-y)\;dx\,dy < \infty\end{aligned}$$

by hypothesis on P_0 and definition of ω. We conclude, in particular, that the operator $\widetilde{P}$ on $L^2([0,1)^n,\mathbb{R}^n)$ defined by $f \mapsto \int_{[0,1)^n} \widetilde{p}(x,y)f(y)dy$ is bounded.

The next four lemmas are extensions of corresponding Lemmas 4-7 in section I.2 of [2] to the setting of $\mathcal{H}^2_\omega$ versus L^2_ω. In the next lemma, even though the result will be applied to functions in $\mathcal{H}^2$, the estimates just depend on the size of the functions. We set $\widetilde{\mathcal{H}^2} = \{\widetilde{f} = \sum_k f(x-k) : f \in \mathcal{H}^2\}$. Clearly this defines a subspace of $L^2([0,1)^n,\mathbb{R}^n)$. We also set $\mathcal{H}^2_\omega = \mathcal{H}^2 \cap L^2_\omega$ where $L^2_\omega = \{f : \mathbb{R}^n \to \mathbb{R}^n : |f(x)|^2\omega(x) \in L^1(\mathbb{R}^n)\}$. We have:

Lemma 7. *(i) If $f \in \mathcal{H}^2_\omega$ then $\widetilde{f} \in \widetilde{\mathcal{H}^2} \cap L^2([0,1)^n,\mathbb{R}^n)$.*
(ii) If $f \in \mathcal{H}^2_\omega$ then $P_0 f \in \mathcal{H}^2_\omega$
(iii) If $g \in \mathcal{H}^2$ is supported in $[0,1]^n$ then $\widetilde{P_0 g} = \widetilde{P}g$.
(iv) $\widetilde{P}$ projects $\widetilde{\mathcal{H}^2}$ onto $\widetilde{V}_0 = \left\{\widetilde{f} : f \in V_0 \cap \mathcal{H}^2_\omega\right\}$

Proof. To prove (i) we just note that, because ω is a Beurling weight,

$$\begin{aligned}\int_{[0,1)^n}\left|\sum_k f(x-k)\right|^2 dx &\le \int_{[0,1)^n}\sum_k |f(x-k)|^2\,\omega(x-k)\sum_k\frac{1}{\omega(x-k)}dx\\ &\le C\sum_k\frac{1}{\omega(k)}\int_{\mathbb{R}^n}|f(x)|^2\,\omega(x)dx.\end{aligned}$$

To verify (ii) we note that $P_0 f\in\mathcal{H}^2$ because of the divergence-free condition on the kernel. To verify that $P_0f\in L^2_\omega$, using properties of Beurling weights again and shift-invariance of p gives

$$\begin{aligned}&\int\omega(x)\left|\int p(x,y)f(y)dy\right|^2 dx\\ \le\;&\int\omega(x)\left[\sum_k\left(\int_{k+[0,1)^n}|p(x,y)|^2\,dy\right)^{1/2}\left(\int_{k+[0,1)^n}|f(y)|^2\,dy\right)^{1/2}\right]^2 dx\\ \le\;&\int\sum_k\left[\int_{k+[0,1)^n}|p(x,y)|^2\,\omega(x-k)\;dy\right]\left[\int_{k+[0,1)^n}|f(y)|^2\;dy\;\omega(k)\right]\sum_k\frac{\omega(x)}{\omega(x-k)\omega(k)}dx\\ \le\;& C\sum_k\left[\int_{k+[0,1)^n}|f(y)|^2\;dy\;\omega(k)\right]\int_{\mathbb{R}^n}\int_{k+[0,1)^n}|p(x-k,y-k)|^2\,\omega(x-k)\;dy\,dx\\ \le\;& C\int_{\mathbb{R}^n}\int_{[0,1)^n}|p(x,y)|^2\,\omega(x-y)\;dy\,dx\int_{\mathbb{R}^n}|f(y)|^2\,\omega(y)dy\end{aligned}$$

whence the claim follows from the definition of an ω-localized kernel.

To prove (iii), we note that if $g\in\mathcal{H}^2$ is supported in $[0,1]^n$ then $\widetilde{g}=g$ almost everywhere on $[0,1)^n$. To show that $\widetilde{P_0g}=\widetilde{P}g$:

$$\begin{aligned}\widetilde{P}g(x) &= \int_{[0,1)^n}\widetilde{p}(x,y)g(y)\;dy=\sum_l\int_{[0,1)^n}\widetilde{p}(x,y)g(y-l)\;dy\\ &= \sum_l\int_{[0,1)^n}\widetilde{p}(x+l,y)g(y-l)\;dy, \qquad \text{since } \widetilde{p}(\cdot,y) \text{ is periodic}\\ &= \sum_k\sum_l\int_{[0,1)^n}p(x-k+l,y)g(y-l)dy\\ &= \sum_k\sum_l\int_{[0,1)^n}p(x-k,y-l)g(y-l)dy \qquad \text{by shift invariance}\\ &= \sum_k\int_{\mathbb{R}^n}p(x-k,y)g(y)\;dy=\sum_k\left(P_0g\right)(x-k)=\widetilde{P_0g}(x).\end{aligned}$$

Therefore it remains only to prove (iv). First we note that any element of $\widetilde{\mathcal{H}^2}$ defines a periodic, locally square-integrable divergence-free distribution. Hence any such element g can be identified with an element of $\mathcal{H}^2$ supported in $[0,1]^n$, which automatically belongs to $\mathcal{H}^2_\omega$. In view of (iii), therefore, it is enough to show that if $f \in V_0 \cap \mathcal{H}^2_\omega$ then $\widetilde{P}(\widetilde{f}) = \widetilde{f}$. But shift invariance gives:

$$\begin{aligned} \widetilde{P}\widetilde{f}(x) &= \int_{[0,1)^n} \sum_k p(x, y-k) \sum_l f(y-l)\ dy \\ &= \int_{[0,1)^n} \sum_k p(x+k, y) \sum_l f(y-l)\ dy \\ &= \sum_l \int_{[0,1)^n} \sum_k p(x+k-l, y-l) f(y-l)\ dy \\ &= \sum_k \int_{\mathbb{R}^n} p(x+k, y) f(y)\ dy = \sum_k (P_0 f)(x+k) = \widetilde{f}(x) \end{aligned}$$

■

Lemma 8. $\dim \widetilde{V_0} = A$ *where* A *is defined as in Theorem 2.*

Proof. Since $\widetilde{p} \in L^2([0,1)^{2n}, \mathcal{M}_n)$, the operator $\widetilde{P}$ is Hilbert-Schmidt, hence compact. Since the unit ball $\widetilde{B}$ of $\widetilde{V_0}$ is bounded it follows that $\widetilde{P}(\widetilde{B})$ is relatively compact, so that $\dim \widetilde{V_0}$ is finite. Now if $\{\widetilde{f_\alpha}\}$ is a basis of $\widetilde{V_0}$ and $\{\widetilde{f^*_\alpha}\}$ is the dual basis of $\{\widetilde{f_\alpha}\}$ in $\left(\ker \widetilde{P_0}\right)^\perp$ then

$$\widetilde{p}(x,y) = \sum_{\alpha \in \mathcal{A}} \widetilde{f_\alpha}(x) \overline{\widetilde{f_\alpha}}^{*\top}(y) \text{ a.e.}$$

so that

$$\operatorname{tr} \int_{[0,1)^n} \int_{[0,1)^n} [\widetilde{p}(x,y)\widetilde{p}(y,x)]\ \ dy\, dx = \sum_{\alpha \in \mathcal{A}} \left\langle \widetilde{f_\alpha}, \widetilde{f^*_\alpha} \right\rangle = A.$$

This proves the lemma. ■

It still needs to be shown that V_0 *has* a Riesz basis of the desired form. We begin with:

Lemma 9. *Let* $\{f_\alpha\}_{\alpha \in A} \subset L^2(\mathbb{R}^n, \mathbb{R}^n)$*. Then the Gram matrix of* L^2 *inner products of the vector functions* $\{\widetilde{f_\alpha}\}$ *is the autocorrelation matrix* $M_{f_\alpha}(\xi)|_{\xi=0}$.

Proof. A simple computation shows that $\left\langle \widetilde{f_\alpha}, \widetilde{f_{\alpha'}} \right\rangle_{L^2([0,1)^n)} = C(f_\alpha, f_{\alpha'})(0)$. ■

In view of the previous lemma, the main trick to recover properties of V_0 from those of $\widetilde{V_0}$ is to replace the role of $\xi = 0$ by other ξ. In particular, abusing the meaning of the zero in V_0 we define $V_\xi = \{e^{-2\pi i x\cdot\xi} f : f \in V_0\}$. Similarly we set $V_\xi^* = \{e^{-2\pi i x\cdot\xi} f : f \in V_0^*\}$. Then let P_ξ be the projector onto V_ξ parallel to $\left(V_\xi^*\right)^\perp$. Then its kernel $p_\xi(x,y) = p(x,y)e^{-2\pi i\xi\cdot(x-y)}$ is ω-localized. Finally, set $\widetilde{V_\xi} = \{\widetilde{f} : f \in V_\xi \cap e^{-2\pi i x\cdot\xi}\mathcal{H}^2_\omega\}$.

Lemma 10. $\dim \widetilde{V_\xi} = A$ *for all* ξ.

Proof. As P_ξ commutes with integer translations and has an ω-localized kernel, just as before,

$$\dim \widetilde{V_\xi} = \operatorname{tr} \int_{[0,1)^n} \int_{[0,1)^n} \widetilde{p_\xi}(x,y)\widetilde{p_\xi}(y,x)\ dy\, dx = A.$$

This can be seen by direct expansion in terms of p or else from dominated convergence from which we can infer that the integral varies continuously in the parameter ξ. Since it takes integer values it must be constant. ■

Now we construct a Riesz basis of V_0 of the form $\{\phi_\alpha(x-k)\}$. By the previous lemma, for each $\xi \in [0,1)^n$ there exist A functions $\{f_\alpha^\xi\}_{\alpha\in\mathcal{A}}$ in $V_0 \cap H^2_\omega$ such that $\{\left(e^{-2\pi i x\cdot\xi} f_\alpha^\xi\right)^\sim\}_{\alpha\in\mathcal{A}}$ is a basis of $\widetilde{V_\xi}$. But the Gram matrix of $\{\left(e^{-2\pi i x\cdot\xi} f_\alpha^\xi\right)^\sim\}_{\alpha\in\mathcal{A}}$ at $\eta \in [0,1)^n$ is nothing other than the autocorrelation matrix $M_{f_\alpha^\xi}$ evaluated at η. Since $f_\alpha^\xi \in L^2_\omega$, the coefficients of $M_{f_\alpha^\xi}$ have absolutely convergent Fourier series, hence, so does its determinant. Therefore, if $\det M_{f_\alpha^\xi}$ does not vanish at $\eta = \xi$ then it remains minorized by some $\gamma(\xi) > 0$ and the coefficients remain majorized in modulus by $1/\gamma(\xi)$ in a neighborhood $B(\xi, r(\xi))$ of ξ. Then by compactness of $[0,1]^n$ one can extract a finite family $B_\nu = B(\xi_\nu, r_\nu(\xi)) \cap [0,1)^n, 1 \le \nu \le N$ that still covers $[0,1)^n$. Set $C_\nu = B_\nu \setminus \cup_{\mu<\nu} B_\mu$ and let D_ν be the union of all integer shifts of C_ν. Now define

$$\widehat{\phi}_\alpha(\xi) = \sum_{\nu=1}^N \widehat{f_\alpha^{\xi_\nu}} \chi_{D_\nu}(\xi).$$

We must show that $\phi_\alpha \in V_0$. It suffices to show that ϕ_α is orthogonal to each element of the orthogonal complement of V_0. First, $\widehat{f_\alpha^{\xi_\nu}}\chi_{D_\nu} \in \widehat{\mathcal{H}^2}$ because χ_{D_ν} is bounded and scalar-valued. It follows that if $g \in (V_0)^\perp$ then, by periodicity of χ_{D_ν},

$$\begin{aligned} C(\left(\widehat{f_\alpha^{\xi_\nu}}\chi_{D_\nu}\right)^\vee, g)(\xi) &= \chi_{D_\nu}(\xi) C(f_\alpha^{\xi_\nu}, g)(\xi) \\ &= \chi_{D_\nu}(\xi) \sum_{k\in\mathbb{Z}^n} \left\langle f_\alpha^{\xi_\nu}(\cdot + k), g\right\rangle e^{-2\pi i k\cdot\xi} = 0 \end{aligned}$$

because $\left\langle f_\alpha^{\xi_\nu}(\cdot + k), g\right\rangle = 0$ for all k. One concludes from this that $\left\langle (\widehat{f_\alpha^{\xi_\nu}}\chi_{D_\nu})^\vee, g\right\rangle = 0$ and, therefore, each $\phi_\alpha \in V_0$. Next, since the sets D_ν are disjoint,

$$C(\phi_\alpha, \phi_{\alpha'}) = \sum_{\nu,\nu'=1}^{N} \chi_{D_\nu}\chi_{D_{\nu'}} C(f_\alpha^{\xi_\nu}, f_{\alpha'}^{\xi_{\nu'}}) = \sum_{\nu=1}^{N} \chi_{D_\nu} C(f_\alpha^{\xi_\nu}, f_{\alpha'}^{\xi_\nu}).$$

Hence the coefficients of the autocorrelation matrix of the $\{\phi_\alpha\}$ remain majorized by $\max_{1\le\nu\le N}\{1/\gamma(\xi_\nu)\}$ and the determinant remains bounded below by $\min_{1\le\nu\le N}\{\gamma(\xi_\nu)\}$. By Lemma 5, therefore, the $\{\phi_\alpha(\cdot - k)\}_{\alpha\in\mathcal{A}, k\in\mathbb{Z}^n}$ form a Riesz family in V_0.

It just remains to show that the $\{\phi_\alpha\}$ generate all of V_0. But, if $f \in V_0 \cap \mathcal{H}^2_\omega$ then $(e^{-2\pi i x\cdot\xi} f)^\sim$ can be expressed as a linear combination of $\left(e^{-2\pi i x\cdot\xi} f_\alpha^{\xi_\nu}\right)^\sim$ whenever $\xi \in C_\nu$, where the coefficients are bounded. In fact, if we call these coefficients $r_{\alpha,\nu}(\xi)$, they satisfy

$$M_{f_\alpha^{\xi_\nu}}|_\xi \cdot \overrightarrow{r}_{\alpha,\nu}(\xi) = \overrightarrow{C}(f_\alpha^{\xi_\nu}, f)$$

where the vector index runs over $\mathcal{A}$. Now if $R_{\alpha,\nu}(\xi) = \sum_k r_{\alpha,\nu}(\xi + k)$ then $\widehat{f}\chi_{D_\nu} = \sum_\alpha R_{\alpha,\nu}(\xi)\chi_{D_\nu}\widehat{f_\alpha^{\xi_\nu}}$ so that, finally,

$$\widehat{f} = \sum_\alpha \left(\sum_\nu R_{\alpha,\nu}(\xi)\chi_{D_\nu}(\xi)\right)\widehat{\phi_\alpha}$$

which proves that the $\{\phi_\alpha(x-k)\}$ are complete in V_0. Since $V_0 \cap \mathcal{H}^2_\omega = P_0(\mathcal{H}^2_\omega)$ and $\mathcal{H}^2_\omega$ is dense in $\mathcal{H}^2$ it follows that $V_0 \cap \mathcal{H}^2_\omega$ is dense in V_0. Hence $\{\phi_\alpha(x-k)\}$ form a Riesz basis for V_0. This completes the proof of Theorem 2.

3.3. Proof of Theorem 3.

Lemma 11. *If the finite collection $\{\phi_\alpha\}_{\alpha\in\mathcal{A}}$ has compact support then the correlation matrix $M_{\phi_\alpha}(\xi)$ of $\{\phi_\alpha\}$ takes values in trigonometric polynomials.*

Proof. Given any pair of square integrable compactly supported functions their correlation function is a trigonometric polynomial. This follows from expanding the correlation function in a Fourier series and using Parseval's formula. ■

Before getting to the final point, we need a little more notation. Let V be a FSI subspace of $\mathcal{H}^2$ that possesses an ω-localized kernel so that, in particular, Theorem 2 holds. Let $\{\varphi_\beta\}_{\beta\in\mathcal{B}}$ be any finite subset of V,with $|\mathcal{B}| = B$.

Lemma 12. *The collection of vectors $\left\{\{\widehat{\varphi}_\beta(\xi+k)\}_{k\in\mathbb{Z}^n} : \beta\in\mathcal{B}\right\} \subset l^2(\mathbb{Z}^n, \mathbb{C}^n)$ is linearly independent if and only if the correlation matrix $M_{\varphi_\beta}(\xi)$ is nonsingular at ξ.*

Proof. Suppose that $\left\{\widehat{\varphi}_\beta(\xi+k)\right\}_k$, $\beta \in \mathcal{B}$ are linearly dependent. Then there are constants a_β, not all zero, such that $\sum_\beta a_\beta \widehat{\varphi}_\beta(\xi+k) = 0$. This implies that $0 = C(\sum_\beta a_\beta \varphi_\beta, \varphi_\alpha)(\xi) = \sum_\beta a_\beta C(\varphi_\beta, \varphi_\alpha)(\xi)$. Therefore the rows of $M_{\varphi_\beta}(\xi)$ are linearly dependent. Conversely, suppose that the rows of $M_{\varphi_\beta}(\xi)$ are linearly dependent. Then, as before, there is a nontrivial solution of $0 = C(\sum_\beta a_\beta \varphi_\beta, \varphi_\alpha)(\xi)$. Therefore the vector $\left\{\sum_\beta a_\beta \widehat{\varphi}_\beta(\xi+k)\right\}_k$ is orthogonal to each $\left\{\widehat{\varphi}_\beta(\xi+k)\right\}_k$. This can only happen if $\left\{\sum_\beta a_\beta \widehat{\varphi}_\beta(\xi+k)\right\}_k$ is the zero vector. In particular the vectors $\left\{\widehat{\varphi}_\beta(\xi+k)\right\}_k$ are linearly dependent. ∎

The proof of Theorem 3 is simple now. Let $\{\varphi_\beta\}_{\beta\in\mathcal{B}}$ be a compactly supported family in V_0 that is maximal with respect to the property that $\det M_{\varphi_\beta} = 0$ only on a set of measure zero. Then $\{\varphi_\beta\}_{\beta\in\mathcal{B}}$ must be a finite set. In fact, $\{\varphi_\beta(\cdot-k)\}$ forms a basis for the compactly supported elements of V_0. Otherwise, there would be a compactly supported element $\varphi_0 \in V_0$ independent of these. Look at the correlation matrix of $\{\varphi_0, \varphi_\beta\}_{\beta\in\mathcal{B}}$. Its determinant is a trigonometric polynomial, hence either vanishes identically or vanishes on a set of measure zero. The latter case would contradict maximality of $\{\varphi_\beta\}_{\beta\in\mathcal{B}}$. The former case implies that the rows of the correlation matrix are linearly dependent, which is tantamount to saying that $C(\varphi_0,\varphi_\beta)(\xi) = \sum a_{\beta'}(\xi) C(\varphi_{\beta'}, \varphi_\beta)(\xi)$ where the functions $a_{\beta'}(\xi)$ are trig polynomials as well. As we saw above, this amounts to saying that $\{\widehat{\varphi}_0(\xi+k)\}_k \equiv \sum_{\beta,l} a_{\beta,l} e^{-2\pi i l\cdot\xi} \left\{\widehat{\varphi}_\beta(\xi+k)\right\}_k$ where $a_{\beta,l} \neq 0$ for only finitely many $l \in \mathbb{Z}^n$. But this implies that $\varphi_0(x) = \sum_{\beta,l} a_{\beta,l}\varphi_\beta(x-l)$, which contradicts independence of $\{\varphi_0, \varphi_\beta(\cdot-k)\}_{\beta\in\mathcal{B}}$. Therefore, the compactly supported elements of V_0 are contained in the span of $\{\varphi_\beta(\cdot-k)\}$. Since the compactly supported elements of V_0 are dense in V_0, it follows that V_0 is the closure of the span of $\{\varphi_\beta(\cdot - k\}_{\beta\in\mathcal{B},k\in\mathbb{Z}^n}$. This completes the proof of Theorem 3. ∎

4. Nonexistence

To complete the proof that no compactly supported divergence-free wavelet bases exist, we need to apply the results above to the special case where the space in question is generated by the long-scale wavelets. In particular, the fact that the kernel is ω-localized should be a simple corollary of the compact support of the wavelets:

Lemma 13. *Under the hypotheses of Theorem 1, the projector P_0 onto the subspace V_0 of $\mathcal{H}^2$ generated by $\{\psi_\lambda\}_{\lambda\in\Lambda,|Q(\lambda)|>1}$ has an ω-localized kernel.*

Proof. First, because of biorthogonality it is clear that the kernel of P_0 should have the form

$$p(x,y) = \sum_{\lambda\in\Lambda,|Q(\lambda)|>1} \psi_\lambda(x)\overline{\psi_\lambda^{*\top}}(y).$$

The fact that the components of this matrix-valued kernel are ω-localized follows from precisely the same argument as in the scalar case. In particular, one can show that $|p(x,y)|_{\mathcal{M}_n} \leq C(1+|x-y|)^{-n-\gamma}$ where $\gamma > 0$ depends on the Hölder regularity of the wavelets. But then p is ω-localized with respect to $\omega = (1+|x|)^{n+\rho}$ whenever $0 < \rho < 2\gamma$. ■

Corollary 14. *Under the same hypotheses, the compactly supported elements of V_0 are dense in V_0.*

Proof. Clearly finite linear combinations of $\{\psi_\lambda\}_{\lambda\in\Lambda,|Q(\lambda)|>1}$ are compactly supported. But if $f \in V_0$ then its wavelet coefficient sequence $\{\langle f, \psi^*_\lambda\rangle\}_{\lambda\in\Lambda,|Q(\lambda)|>1}$ is square summable, which implies that f can be approximated in V_0 by compactly supported elements. ■

Now, applying Theorem 3 gives:

Corollary 15. *V_0 is locally finite dimensional.*

Proof. By Theorem 3, there is a finite collection of compactly supported functions $\{\phi_\beta\} \subset V_0$ such that V_0 is the l^2-closed span of $\{\phi_\beta(x-k)\}$. In particular, the restriction of any $f \in V_0$ to a compact set K is a linear combination of the restrictions to K of the $\{\phi_\beta(x-k)\}$, and only finitely many of these do not vanish identically on K. ■

Now we can use the corollary to obtain a contradiction that proves Theorem 1. Assume that $V_0 = \oplus_{-\infty}^{-1} W_j = \mathcal{H}^2 \ominus (\oplus_0^\infty W_j)$ where W_j is generated by the families $\{\psi_\lambda : |Q(\lambda)| = 2^{-nj}\}$, where the ψ_λ and the dual wavelets $\psi^*_{\lambda'}$ are both divergence-free and compactly supported. Let h be a harmonic function on $\mathbb{R}^n$ so that ∇h is divergence free. Set $h_M = \nabla h \chi_{\{|x|<M\}}$ where M is chosen such that if ψ^*_λ has support intersecting $[0,1)^n$ and $|Q(\lambda)| \leq 1$ then ψ^*_λ is supported in $\{|x| < M\}$. For such ψ^*_λ it follows that

$$\int_{\mathbb{R}^n} h_M \cdot \psi^*_\lambda = \int_{\mathbb{R}^n} \nabla h \cdot \psi^*_\lambda = -\int_{\mathbb{R}^n} h\,(\nabla \cdot \psi^*_\lambda) = 0$$

by integration by parts together with the fact that $\nabla \cdot \psi^*_\lambda = 0$.

This computation shows that all wavelets ψ^*_λ such that $|Q(\lambda)| \leq 1$ having support intersecting $[0,1)^n$ vanish as linear functionals against ∇h. We conclude that the restriction of $P_0(\nabla h \chi_{\{|x|<M\}})$ to $[0,1)^n$ agrees with the restriction of ∇h to $[0,1)^n$. But the previous results show that restrictions of V_0 to $[0,1)^n$ define a finite dimensional space, whereas restrictions of ∇h to $[0,1)^n$ where h is harmonic form an infinite dimensional space. This contradiction proves Theorem 1. ■

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