ON THE TWO WEIGHTS PROBLEM FOR THE HILBERT TRANSFORM

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§0 INTRODUCTION

In this paper, we prove sufficient conditions on pairs of weights (u, v) (scalar, matrix or operator valued) so that the Hilbert transform

$$Hf(x) = \text{p. v.} \int \frac{f(y)}{x-y} dy,$$

is bounded from $L^2(u)$ to $L^2(v)$. When $u = v$ are scalar, the classical results were given in [HMW] and [CF]. Earlier, [HS] gave a characterization of these weights by complex methods which has been generalized by [CS1] and [CS2] to the case of unequal weights. However these complex-analytic results give conditions which as stated by [CS2] "are not susceptible of being verified in practice." What follows shall be all in the category of real analysis. Matrix results for equal weights have recently been given in [TV]. For u and v scalar weights, a different sufficient condition from ours was given in [F]. More general conditions than ours for the scalar case have recently been given by [TVZ] using very different methods which do not seem to generalize to the operator valued case.

We shall consider only (u, v) so that $u^{-1}, v \in L^{1+\epsilon}_{loc}$ are positive and $u^{-1}, u^{-(1+\epsilon)}, v, v^{1+\epsilon}$ are doubling. There will be an auxiliary Hilbert space $\mathcal{H}$, with scalar product denoted by $\langle \cdot, \cdot \rangle_{\mathcal{H}}$. The weights u and v shall be operator valued and we define for $\mathcal{H}$ valued functions f,

$$\|f\|^2_{L^2(u)} = \int \langle u(x) f(x), f(x) \rangle_{\mathcal{H}} dx.$$

Then we shall prove the following theorem:

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Theorem. *If (u, v) as above satisfy conditions (a),(b), and (c) then*

$$H : L^2(u) \longrightarrow L^2(v).$$

For a full description of conditions (a),(b), and (c) in the scalar and matrix cases see section 3. We briefly describe the conditions here.

Condition (a) will state that for certain Haar multipliers M_u and M_v, the operators $u^{-\frac{1}{2}}M_u^{\frac{1}{2}}$ and $v^{\frac{1}{2}}M_v^{-\frac{1}{2}}$ are bounded on $L^2(\mathbb{R}, \mathcal{H})$. Operators of this form were first studied in [P1]. They were first used to study boundedness of the Hilbert transform in [TV]. In section 1, we describe sufficient conditions for their boundedness in the scalar case. The weakness of these conditions, and their relation to the classical A_p conditions on weights make condition (a) seem reasonable.

Condition (b) is a sort of non-local A_2 condition for $(v^{1+\epsilon}, u^{-(1+\epsilon)})$. Condition (c) is the boundedness of two weighted paraproducts. (In the operator case, part of condition (c) is also a seemingly slightly stronger assumption - an inequality that in the scalar case automatically follows. We point out that in the setting of [TV], this inequality may be replaced by the reverse inequality to A_2. I.e., the inequality:

$$(\frac{1}{|I|}\int_I w^{-1})^{\frac{1}{2}}(\frac{1}{|I|}\int_I w)(\frac{1}{|I|}\int_I w^{-1})^{\frac{1}{2}} \geq C,$$

which in the scalar case simply follows from Hölder's inequality.)

In the matricial case when $u = v$, conditions (a), (b), and (c) are equivalent to the classical Muckenhoupt A_2 condition.

Our theorem should be thought of as a sort of $T(1)$ theorem (see [D]) for two weights. In particular, condition (c) should be seen as the analogue of requiring that $T(1)$ and $T^*(1)$ are in BMO. In this way, our proof differs from that of [TV] in the case that the weights are equal. We use only the standard kernel properties of the Hilbert transform H, namely the decay of matrix coefficients H_{IJ} when $3I \cap 3J = \emptyset$ and the general decay of Hh_I. Further, we prove our bounds using not the Senechkin-Vinogradov test (as in [TV]) but rather the two fundamental lemmas of linear algebra:

Lemma 0.1(Cotlar). *Let T_j be operators on $\mathcal{H}$, a Hilbert space. Suppose that for any j, k, one has*

$$\|T_iT_j^*\|_{\mathcal{H}\to\mathcal{H}} \leq a(|i-j|),$$

and

$$\|T_j^*T_i\|_{\mathcal{H}\to\mathcal{H}} \leq a(|i-j|),$$

where $\sum_j a(j)^{\frac{1}{2}} \leq C$ then

$$\|\sum_{j=-N}^{j=N} T_j\|_{\mathcal{H}\to\mathcal{H}} \leq C,$$

with constant independent of N*.*

For a proof see [D]. Decomposition of an operator T into $\sum_j T_j$ with the T_j's satisfying the hypotheses of the Lemma is called Cotlarization. The other fundamental lemma of linear algebra (in the scalar case), is

Lemma 0.2 (Schur). *Let* T *be an operator on* $L^2(X)$ *with* X *a measure space and let* $K(x,y)$*, its scalar-valued kernel be positive. Suppose there are postive functions* $w_1(x)$ *and* $w_2(x)$ *with*

$$\int w_1(x)K(x,y)dx \le C_1 w_2(y),$$

and

$$\int w_2(y)K(x,y)dy \le C_2 w_1(x).$$

Then $\|T\|_{L^2(X)\to L^2(X)} \le (C_1C_2)^{\frac{1}{2}}$*.*

A proof may be found in [Da]. We state and prove a version in the operator case, (Lemma 3.1), which, while it is not deep, we have been unable to locate in the literature in this form.

Finally, we remark that the most important problem in the field of weighted norm inequalities for the Hilbert transform is to find the necessary and sufficient condition when $u = v$ in the case that $\mathcal{H}$ is not finite dimensional. It is conjectured that the condition is A_2. We do not know whether all A_2 weights satisfy our sufficient conditions, since the generalization of Gehring's theorem [G] is unclear. Also unclear is the correct definition for Carleson condition. We hope our paper inspires future work.

§1 Carleson conditions and bounded operators

We let $\mathcal{D}$ denote the set of dyadic intervals in the real line. We say that a sequence of real numbers $\{b_I\}$ indexed by $\mathcal{D}$ is a Carleson sequence provided that for any $I \in \mathcal{D}$, we have that

$$\sum_{J\in\mathcal{D}, J\subset I} b_J^2 \le C|I|.$$

We recall the Carleson Lemma. (For a proof see [M], p.261.)

Lemma 1.1 (Carleson). *Let* λ_I *be any sequence of real numbers. Define the function*

$$\lambda^*(x) = \sup_{x\in I} |\lambda_I|.$$

Then

$$|\sum_I \lambda_I b_I^2| \le C \int \lambda^*(x)dx.$$

For any interval I, we define h_I to be the Haar function of I,

$$h_I(x) = \frac{1}{\sqrt{|I|}}(\chi_{I^l} - \chi_{I^r}),$$

where I^l and I^r are the left and right children of I, the function χ_J for any interval J is the characteristic function of J, and $|I|$ denotes the length of I. The h_I's form an orthonormal basis of $L^2(\mathbb{R})$. To any sequence b_I, we associate an operator π_b, its paraproduct by

$$\pi_b f = \sum_{I\in\mathcal{D}} b_I h_I m_I f,$$

where $m_I f = \frac{1}{|I|}\int_I f$ is the mean of f on I. (More commonly, π_b is referred to as the paraproduct with or by the BMO function $b = \sum_{I\in\mathcal{D}} b_I h_I$. However throughout this paper the sequences $\{b_I\}$ occur far more naturally than the function b and we prefer to think of π_b as an operator associated to the sequence rather than as a modified product with the function.)

Corollary 1.2. *The operator π_b is bounded on $L^2(\mathbb{R})$ if and only if b is a Carleson sequence*

Proof. For one direction, we simply compute that if b is a Carleson sequence,

$$\|\pi_b f\|^2_{L^2(\mathbb{R})} = \sum_I b_I^2 m_I f^2 \le C\int \mathcal{M}f^2 \le C\|f\|^2_{L^2(\mathbb{R})}.$$

The first inequality follows from Lemma 1.1, where $\mathcal{M}f$ denotes the dyadic maximal function of f. The second inequality follows from the $L^2(\mathbb{R})$ boundedness of the dyadic maximal function, see [D]. On the other hand, if π_b is bounded then $\|\pi_b\chi_I\|^2_{L^2(\mathbb{R})} \le C|I|$. However

$$\|\pi_b\chi_I\|^2_{L^2(\mathbb{R})} \ge C\sum_{J\subset I} b_J^2.$$

Hence b is a Carleson sequence. □

Throughout this section, v shall be a weight - that is - a nonnegative L^1_{loc} function, and u_I and b_I shall be sequences indexed by intervals (all intervals in the remainder of this paper shall be dyadic). We shall concern ourselves with two kinds of operators

$$T_{v,u}f = v\sum_I \frac{\langle f, h_I\rangle}{u_I} h_I = vM_u^{-1}f \tag{1.1}$$

and

$$S_{v,u,b}f = M_u^{-1}\pi_b(vf). \tag{1.2}$$

Here, obviously, M_u denotes the Haar multiplier with coefficients u_I, and $\langle \cdot, \cdot \rangle$ denotes the scalar product in $L^2(\mathbb{R})$. In this section, following [TV], we shall show that the L^2 boundedness of the operators in (1.1) and (1.2) is related. We shall give sufficient conditions, and we shall demonstrate their relationship with the now classical Coifman-Muckenhoupt conditions on weights (see [CF]).

It is clear that a necessary condition for $T_{v,u}$ to be bounded is that $m_I(v^2) \leq C u_I^2$. Let $b_I = \frac{\langle v^2, h_I \rangle}{m_I(v^2)}$. Then we have:

Proposition 1.3. *If $m_I(v^2) \leq C u_I^2$ then $T_{v,u}$ is bounded on $L^2(\mathbb{R})$ if and only if $S_{v,u,b}$ is bounded on $L^2(\mathbb{R})$.*

Proof. First observe that since $m_I(v^2) \leq C u_I^2$, we have that $S^*_{v,u,b} h_I = \frac{\langle v^2, h_I \rangle v \chi_I}{u_I m_I(v^2)|I|}$ is a bounded set in $L^2(\mathbb{R})$. Hence, $g_I = T_{v,u} h_I - S^*_{v,u,b} h_I$ is a bounded set in $L^2(\mathbb{R})$. If g_I is also an orthogonal set in $L^2(\mathbb{R})$, then $T_{v,u} - S^*_{v,u,b}$ is a bounded operator on $L^2(\mathbb{R})$. Which would prove the proposition.

But in fact g_I is an orthogonal set. To see this observe that for each interval I, the function g_I is supported on the interval I and that restricted to each of the left and right halves of I, it is a constant multiple of v. Thus to show that g_I is an orthogonal set, it suffices to show that $g_I \perp v \chi_I$. But this is easy to verify since

$$\int g_I v \chi_I = \frac{1}{u_I} \Big(\int v^2 h_I - \frac{\langle v^2, h_I \rangle}{|I| m_I v^2} \int_I v^2 \Big) = 0,$$

which proves the proposition. □

Next, we give a sufficient condition for $S_{v,u,b}$ to be bounded.

Proposition 1.4. *Suppose there exists $\delta > 0$ so that $\Big(\frac{(m_I(v^{2+\delta}))^{\frac{1}{2+\delta}}}{u_I}\Big) b_I$ is a Carleson sequence. Then the operator $S_{v,u,b}$ is bounded on $L^2(\mathbb{R})$.*

Proof. We have that

$$\|S_{v,u,b} f\|_{L^2}^2 = \sum_I \frac{b_I^2}{u_I^2} \big(m_I(vf)\big)^2.$$

However by Hölder's inequality,

$$m_I(vf) \leq \big(m_I(v^{2+\delta})\big)^{\frac{1}{2+\delta}} \big(m_I(f^{\frac{2+\delta}{1+\delta}})\big)^{\frac{1+\delta}{2+\delta}}.$$

Now simply applying Carleson's lemma and the boundedness of the dyadic maximal function on $L^{\frac{2+2\delta}{2+\delta}}$ proves the proposition. □

Corollary 1.5. *Suppose $w \in \mathrm{RH}_2$, that is there exists a constant C so that for any dyadic interval I, $m_I(w^2) \leq C(m_I w)^2$. Then $T_{w,m_I w}$ is bounded on $L^2(\mathbb{R})$.*

Proof. If $w \in \mathrm{RH}_2$ then $w^2 \in A_\infty$. Hence, $\frac{\langle w^2, h_I\rangle}{m_I(w^2)} = b_I$ is a Carleson sequence [FKP]. By Proposition 1.3, we need only show that $S_{w,m_I w,b}$ is bounded, but this follows immediately from the fact that for some $\delta \geq 0$, we have that $w \in \mathrm{RH}^{2+\delta}$ together with Proposition 1.4. $\square$

For other proofs, applications, and L^p versions of Corollary 1.5, see ([P1],[P2],[KP]).

Corollary 1.6. *Suppose that $w \in A_2$. Then the operators $T_{w^{\frac{1}{2}},(m_I w)^{\frac{1}{2}}}$ and $T_{w^{-\frac{1}{2}},(m_I w)^{-\frac{1}{2}}}$ are bounded on $L^2(\mathbb{R})$.*

Proof. By Propositions 1.3 and 1.4, the operator $T_{w^{\frac{1}{2}},(m_I w)^{\frac{1}{2}}}$ is bounded for any $w \in A_\infty$. This follows from $\frac{\langle w, h_I\rangle}{m_I w}$ being a Carleson sequence, which occurs when $w \in A_\infty$, as well as the fact that $w \in \mathrm{RH}^{1+\epsilon}$, for some $\epsilon > 0$. Now since $w^{-1} \in A_\infty$, we have that $T_{w^{-\frac{1}{2}},(m_I(w^{-1}))^{\frac{1}{2}}}$ is bounded. But since $w \in A_2$, it is the case that $\frac{1}{m_I w} \geq m_I(w^{-1})$. This together with the boundedness of Haar multipliers with bounded coefficients proves the corollary. $\square$

We remark that Corollary 1.6 gives a trivial proof of the boundedness of Haar multipliers with bounded coefficients on $L^2(w)$ for any $w \in A_2$. The corollary says that $w^{\frac{1}{2}} M_w^{-\frac{1}{2}}$ and the adjoint of its inverse $w^{-\frac{1}{2}} M_w^{\frac{1}{2}}$ are bounded where M_w is the Haar multiplier with coefficients $m_I w$. Let L be a Haar multiplier with bounded coefficients. Then it is bounded on $L^2(w)$ if and only if $w^{\frac{1}{2}} L w^{-\frac{1}{2}}$ is bounded on $L^2(\mathbb{R})$. By the boundedness of the operators from Corollary 1.6 and their adjoints, this is true if and only if $M_w^{\frac{1}{2}} L M_w^{-\frac{1}{2}}$ is bounded on $L^2(\mathbb{R})$. But everything commutes and $M_w^{\frac{1}{2}} L M_w^{-\frac{1}{2}} = L$. Hence L is bounded on $L^2(w)$.

Similarly, one has a simple proof that π_b where b is a Carleson sequence is bounded on $L^2(w)$ when $w \in A_2$. We simply observe that it suffices to show that $M_w^{\frac{1}{2}} \pi_b w^{-\frac{1}{2}}$ is bounded on $L^2(\mathbb{R})$. Now we apply Proposition 1.4 using the fact that $w \in A_2$ implies $w \in A_{2-\delta}$.

The same ideas can be used to give simple sufficient conditions for L and π_b to satisfy two weight inequalities. For example, L takes $L^2(u) \longrightarrow L^2(v)$ provided there exist sequences c_1 and c_2 so that $T_{u^{-\frac{1}{2}},c_1}$ and $T_{v^{\frac{1}{2}},c_2}$ are bounded and (c_1, c_2) satisfy an A_2 condition, i.e $\frac{1}{c_{1,I} c_{2,I}} \leq C$ for every dyadic I. Similarly, π_b is bounded from $L^2(u)$ to $L^2(v)$ provided $T_{v^{\frac{1}{2}},c_2}$ is bounded and there exists $\delta > 0$ so that $\left(m_I(v^{1+\frac{\delta}{2}})\right)^{\frac{1}{2+\delta}} \leq C c_{2,I}$. These ideas form the basis for our two weights result for the Hilbert transform. First, however, we discuss the relationship of the boundedness of $T_{v^{\frac{1}{2}},c_2}$ to $v \in A_\infty$.

As mentioned in the proof of Corollary 1.6, for any $w \in A_\infty$, we have that $T_{w^{\frac{1}{2}},(m_I w)^{\frac{1}{2}}}$ is bounded. This followed from the fact that $w \in \mathrm{RH}^{1+\epsilon}$, for some $\epsilon > 0$. In what follows, define for any ϵ, $w_{I,\epsilon} = \big(m_I(w^{1+\epsilon})\big)^{\frac{1}{1+\epsilon}}$. Fixing $\epsilon > 0$, we may ask when $T_{w^{\frac{1}{2}},(w_{I,\epsilon})^{\frac{1}{2}}}$ is bounded. Propositions 1.3 and 1.4 give as a sufficient condition that there exists a $\delta > 0$ so that

$$b_I = \frac{\langle w, h_I\rangle (w_{I,\delta})^{\frac{1}{2}}}{(m_I w)(w_{I,\epsilon})^{\frac{1}{2}}},$$

is a Carleson sequence. By Hölder's inequality, it is certainly sufficient that $c_I = \frac{\langle w, h_I\rangle}{w_{I,\mu}}$ is a Carleson sequence provided that $0 < \mu < \frac{\epsilon}{2+\epsilon}$. We do not necessarily get the result when $\mu = \frac{\epsilon}{2+\epsilon}$ since weights not in A_∞ do not necessarily satisy a reverse Hölder condition. When c_I is a Carleson sequence we say that $w \in A_{\infty+\mu}$. Certainly, if $w \in A_\infty$ then $w \in A_{\infty+\mu}$. A priori, one might believe that any weight in $A_{\infty+\mu}$ is in A_∞ or that all weights are in $A_{\infty+\mu}$. In fact, neither is the case. We thank Peter Jones for the following examples.

First consider $w(x) = |\frac{(\log x)^{-2}}{x}|$ on the interval $[0, 1/2]$. We have that $w(x)$ is not in A_∞ since it is not in $L^{1+\mu}$ for any μ. However, on every interval not containing 0, it satifies a reverse Hölder inequality with uniform estimates. Thus we have $w \in A_{\infty+\mu}$ since on those intervals I containing 0, we have $c_I = 0$ while on the others we simply apply [FKP]. In other words, to sum c_I^2 for I's contained in an interval of the form $[0, 2^{-j}]$, we need only sum it over intervals contained in intervals of the form $[2^{-k-1}, 2^{-}k]$ with $k > j$, apply [FKP] to each of these and sum the geometric series.

Next, we define a weight w_j with the parameter j an integer. The $A_{\infty+\mu}$ constant for any $\mu > 0$ will be unbounded as we vary the parameter j. We define $f_{j,[0,1]}$ to be the function defined on the interval $[0, 1]$ which takes on the value $2^j - \delta$ on $[0, 2^{-j}]$, is constant on the rest of [0,1] and has mean 1. For any interval I, we let $f_{j,I}$ be the same function rescaled to the interval I. Let $w_{j,0} = f_{j,[0,1]}$ We choose $\delta << \frac{1}{(2^j)^{(2^j)^{2^j}}}$, so that we may neglect it for what follows. Now we define $w_{j,1}$ by letting it equal $w_{j,0}$ in the interval $[0, 2^{-j}]$. Now in the interval $[k2^{-j}, (k+1)(2^{-j})]$ for $1 \le k \le 2^{j-1}$ we let $w_{j,1} = w_{j,0} f_{j,[k2^{-j},(k+1)(2^{-j})]}$. We repeat the procedure 2^j times letting $w_j = w_{j,2^j}$. Now

$$r_j = 2^j \sum_{[0,2^{-j}] \subset J \subset [0,1]} \frac{\langle f_{j,[0,1]}, h_J\rangle^2}{f_J^2},$$

can readily be seen to be comparable to the $A_{\infty+\epsilon}$ constant of w_j when

$$f_J = \big(\frac{1}{|J|}\int_J (f_{j,[0,1]})^{1+\epsilon}\big)^{\frac{1}{1+\epsilon}}.$$

But r_j is readily seen to be approximately $2^{\frac{j}{1+\epsilon}}$.

§2 A small section on operators

The purpose of this section is just to discuss the generalizations of Jensen's inequality and Schur's lemma which we shall be using in the proof of the main theorem.

From this point on, $\mathcal{H}$ will be a Hilbert space. We will think of H, the Hilbert transform as acting on $L^2(\mathbb{R}, \mathcal{H})$, the space of square integrable Hilbert space valued functions. This space is the same as $L^2(\mathbb{R})\otimes\mathcal{H}$. Naturally, we define the action of H by $H(f\otimes v) = (Hf\otimes v)$. Our weights u and v shall be positive operator valued functions on $\mathcal{H}$. For any two self-adjoint operators A and B, we say that $A \leq B$ when $B - A$ is positive; and for C a constant, $A \leq C$ means that $(C\,Identity - A)$ is positive.

First, we state and prove the correct version of Schur's Lemma for operator valued kernels.

Lemma 2.1 (Schur). *Let X be a measure space. And let $K(x,y)$ be a $\mathcal{B}(\mathcal{H})$ valued function on $X \times X$. Suppose that $K(x,y) = A(x,y)B(x,y)$ where the multiplication is pointwise composition. Suppose further that*

$$\int A(x,y)A^*(x,y)dy \leq C_1,$$

and that

$$\int B^*(x,y)B(x,y)dx \leq C_2.$$

Then $K(x,y)$ gives rises to a bounded operator on $L^2(\mathbb{R}, \mathcal{H})$ with bound $\sqrt{C_1C_2}$.

Proof. We need only bound $\int\langle f(x), K(x,y)g(y)\rangle_{\mathcal{H}}dxdy$. We observe:

$$\int \langle f(x), A(x,y)B(x,y)g(y)\rangle_{\mathcal{H}}dxdy \leq \int \langle A^*(x,y)f(x), B(x,y)g(y)\rangle_{\mathcal{H}}dxdy$$

$$\leq \Big(\int |A^*(x,y)f(x)|^2dxdy\Big)^{\frac{1}{2}}\Big(\int |B(x,y)g(y)|^2dxdy\Big)^{\frac{1}{2}};$$

here $|\cdot|$ denotes the norm in $\mathcal{H}$, i.e. $\|\cdot\|_{\mathcal{H}} = (\langle\cdot,\cdot\rangle_{\mathcal{H}})^{\frac{1}{2}}$.

We write the first integral as

$$\int \langle A(x,y)A^*(x,y)f(x), f(x)\rangle_{\mathcal{H}}dxdy,$$

and bound it by integrating first in y. We do the analogous thing for the second integral. □ □

Further we need to state the operator version of Jensen's inequality.

Lemma 2.2. *Let $A(x)$ be a positive operator valued function on a measure space X. Let $d\mu(x)$ be a measure on X with total measure 1. Let $1 \leq p \leq \infty$. Then $\left(\int A(x)^p d\mu(x)\right)^{\frac{1}{p}} \geq \int A(x) d\mu(x)$.*

For $1 \leq p \leq 2$, the only case in which we will use this, the result follows from [HP] and from the monotonicity of the function $f(t) = t^r$ when $0 \leq r \leq 1$, see ([KR], Exercise 4.6.46. All solutions are provided in [KR2].) Of course, we get immediately by scaling a version of Hölder's inequality:

Lemma 2.3. *Let $A(x)$ be a positive operator valued function and let $f(x)$ be a positive integrable function. Then*

$$\int f(x)A(x)dx \leq \left(\int f(x)\right)^{\frac{1}{q}} \left(\int f(x)A(x)^p dx\right)^{\frac{1}{p}},$$

whenever $1 < p < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. Simply apply Lemma 2.2 to the measure $d\mu(x) = \frac{f(x)dx}{\int f(y)dy}$. $\square$

Many norm estimates will be based on

Lemma 2.4. *Let T_1 and T_2 be positive operators with $T_1 \leq T_2$. Let S be any fixed operator. Then*

$$\|T_1^{\frac{1}{2}} S\| \leq \|T_2^{\frac{1}{2}} S\|.$$

Here $\|.\|$ denotes the operator norm.

This is [KR], exercise 4.6.1.

§3 The two weights problem

In this section we will give a sufficient condition on pairs of doubling weights (u, v) ensuring that the operator $v^{\frac{1}{2}} H u^{-\frac{1}{2}}$ is bounded where H is the Hilbert transform. Here an operator valued weight v is said to be doubling if there exists a constant C so that for any dyadic interval I, whenever $\tilde{I}$ is its parent, one has

$$\int_{\tilde{I}} v \leq C \int_I v,$$

with the inequality in the sense of operators.

As always $\mathcal{D}$ shall denote the set of all dyadic intervals in the real line. The set of dyadic intervals of length 2^{-k} shall be $\mathcal{D}_k$. We shall divide the set of all ordered pairs of dyadic intervals into a union of 5 disjoint sets. Let

$$Z_1 = \{(I, J) : |I| > |J|,\ 3I \cap 3J = \emptyset\},$$

$$Z_2 = \{(I,J) : |I| < |J|,\ 3I \cap 3J = \emptyset\},$$
$$Z_3 = \{(I,J) : |I| < |J|,\ 3I \cap 3J \neq \emptyset\},$$
$$Z_4 = \{(I,J) : |I| > |J|,\ 3I \cap 3J \neq \emptyset\},$$

and

$$Z_5 = \{(I,J) : |I| = |J|\},$$

We let E_I denote the projection onto the Haar function on I, which we denote h_I. In other words, $E_I f = \langle f, h_I \rangle h_I$ for all $f \in L^2(\mathbb{R}, \mathcal{H})$, here $\langle f, h_I \rangle = \int f(x) h_I(x0dx \in \mathcal{H}$. We shall break up the Hilbert transform into corresponding pieces:

$$H_\alpha = \sum_{(I,J)\in Z_\alpha} E_J H E_I.$$

Here α runs from 1 to 5.

We now state our conditions on the pair (u, v) and derive a few easy consequences.

First define $u_I = (\frac{1}{|I|}\int u^{-(1+\epsilon)})^{\frac{1}{1+\epsilon}}$ and $v_I = (\frac{1}{|I|}\int v^{1+\epsilon})^{\frac{1}{1+\epsilon}}$ (we should use $v_{I,\epsilon}$, etc... but to ease the notation we will not). Here the number $\epsilon > 0$ shall be fixed throughout. (By Lemma 2.2, it is clear that if we define the mean $m_I u = \frac{1}{|I|}\int_I u$ then one has the operator inequalities $m_I(u^{-1}) \le u_I$ and $m_I v \le v_I$ which we shall use frequently.)

We define the operators acting on $L^2(\mathbb{R}, \mathcal{H})$

$$T_u f = u^{-\frac{1}{2}} M_u^{-\frac{1}{2}} f = \sum_{I\in\mathcal{D}} u^{-\frac{1}{2}} u_I^{-\frac{1}{2}} \langle f, h_I \rangle h_I,$$

and

$$T_v f = v^{\frac{1}{2}} M_v^{-\frac{1}{2}} f = \sum_{I\in\mathcal{D}} v^{\frac{1}{2}} v_I^{-\frac{1}{2}} \langle f, h_I \rangle h_I.$$

Here M_w^t denotes the Haar multiplier acting on $L^2(\mathbb{R}, \mathcal{H})$ with coefficient w_I^t, for w_I a given sequence of positive selfadjoint operators on $\mathcal{H}$.

We say that (u, v) satisfies condition (a), provided that T_u and T_v are bounded operators on $L^2(\mathbb{R}, \mathcal{H})$.

We say that (u, v) satisfy condition (b) provided that

$$u_I^{\frac{1}{2}}\left[\left(|I|\int_{(3I)^c}\frac{v^{1+\epsilon}}{(x-y_I)^2}\right)^{\frac{1}{1+\epsilon}} + v_I + v_{I-|I|} + v_{I-|I|}\right]u_I^{\frac{1}{2}} \le C, \tag{3.1}$$

and that

$$v_I^{\frac{1}{2}}\left[\left(|I|\int_{(3I)^c}\frac{u^{-(1+\epsilon)}}{(x-y_I)^2}\right)^{\frac{1}{1+\epsilon}} + u_I + u_{I-|I|} + u_{I-|I|}\right]v_I^{\frac{1}{2}} \le C, \tag{3.2}$$

We observe that for any A and B positive operators, writing $B^{\frac{1}{2}}AB^{\frac{1}{2}} \leq C$ with C a constant is the same as writing $\|A^{\frac{1}{2}}B^{\frac{1}{2}}\| \leq C^{\frac{1}{2}}$. We also point out that for any positive operator-valued function w, we have that $\int_{(3I)^c} \frac{w}{|x-y_I|^n} \leq (\frac{1}{|I|^{n-2}}) \int_{(3I)^c} \frac{w}{|x-y_I|^2}$, for $n > 2$. This is not really Hölder's inequality but just the statement that on $(3I)^c$, we have $\frac{1}{|x-y_I|^{n-2}} \leq C\frac{1}{|I|^{n-2}}$.

Finally, we come to condition (c). If c_I is a sequence of bounded operators in $\mathcal{H}$ (not necessarily selfadjoint this time) indexed by the dyadic intervals, we define the paraproduct

$$\pi_c f = \sum_I h_I c_I m_I f.$$

Let $c_I^v = (m_I v)^{-1}\langle v, H_3 h_I\rangle$ and c_I^u the analogous thing for u^{-1}. Then we say that (u, v) satisfy condition (c) provided that $M_u^{\frac{1}{2}}\pi_{c^v}v^{\frac{1}{2}}$ and $M_v^{\frac{1}{2}}\pi_{c^u}u^{-\frac{1}{2}}$ are bounded and that the following inequalities are satisfied for any dyadic $J \subset I$:

$$(m_I v)^{-\frac{1}{2}}\left(\frac{1}{|I|}\int_{I^c} vH_3h_J\right)u_J\left(\frac{1}{|I|}\int_{I^c} vH_3h_J\right)(m_I v)^{-\frac{1}{2}} \leq C\left(\frac{|J|^3}{(d_{IJ})^4}\right)u_I^{-\frac{1}{2}}u_J u_I^{-\frac{1}{2}}. \tag{3.3}$$

and

$$(m_I u^{-1})^{-\frac{1}{2}}\left(\frac{1}{|I|}\int_{I^c} u^{-1}H_3h_J\right)v_J\left(\frac{1}{|I|}\int_{I^c} u^{-1}H_3h_J\right)(m_I u^{-1})^{-\frac{1}{2}} \leq C\left(\frac{|J|^3}{(d_{IJ})^4}\right)u_I^{-\frac{1}{2}}u_J u_I^{-\frac{1}{2}}. \tag{3.4}$$

Here d_{IJ} denotes the distance from J to the boundary of I and the inequalities are in the sense of operators. (By contrast, we will define ρ_{IJ} to be the maximum of $|I|$, $|J|$, and the distance between I and J.)

Let us observe a quick consequence of condition (c). Let $h \in \mathcal{H}$ be a fixed vector. We apply the operator $M_u^{\frac{1}{2}}\pi_{c^v}v^{\frac{1}{2}}$ to the test function $v^{\frac{1}{2}}h\chi_I$. From the Ith summand of this we obtain the size estimate:

$$\|u_I^{\frac{1}{2}}c_I^v(m_I v)^{\frac{1}{2}}\|_{\mathcal{H}\to\mathcal{H}} \leq C|I|^{\frac{1}{2}} \tag{3.5}$$

There is of course an analogous inequality when the u's and v's are switched. In the case in which u and v are scalars, the inequality (3.5) (together with condition (b)) implies the inequality (3.3). We now give the proof of this implication. In the case where v and u are scalar, the definition of c_I^v together with (3.5) implies that

$$\int vH_3h_I \leq C\frac{|I|^{\frac{1}{2}}(m_I v)^{\frac{1}{2}}}{u_I^{\frac{1}{2}}}. \tag{3.6}$$

Now recall that $H_3 h_I$ is constant on intervals whose length is $|I|$. Hence, $|H_3 h_I| \leq C \frac{1}{|I|^{\frac{1}{2}}}$ everywhere and is constant on I. Thus from (3.6),

$$\int_{I^c} v H_3 h_I \leq C\big(g|I|^{\frac{1}{2}} m_I v + \frac{|I|^{\frac{1}{2}}(m_I v)^{\frac{1}{2}}}{u_I^{\frac{1}{2}}}\big). \tag{3.7}$$

But from condition (b) and the fact that $m_I v \leq v_I$, one has that

$$(m_I v)^{\frac{1}{2}} \leq C \frac{1}{u_I^{\frac{1}{2}}}(m_I v u_I)^{\frac{1}{2}} \leq C \frac{1}{u_I^{\frac{1}{2}}}.$$

So that (3.7) implies that

$$\int_{I^c} v H_3 h_I \leq C \frac{|I|^{\frac{1}{2}}(m_I v)^{\frac{1}{2}}}{u_I^{\frac{1}{2}}}.$$

Now we observe that on I^c, the function $H_3 h_I$ is always positive. This is because Hh_I is positive on I^c and $H_3 h_I$ is given by the mean of Hh_I on a certain Whitney decomposition of I^c (we will say more about this in the proof of Theorem 3.1). In fact, we have on I^c that $H_3 h_I(x) \sim \frac{|I|^{\frac{3}{2}}}{\rho_{xI}^2}$. Where we define ρ_{xI} to be the maximum of $|I|$ and the distance from I to x. Thus one has

$$\int_{I^c} H_3 h_J \leq C \frac{|J|^{\frac{3}{2}}|I|^{\frac{1}{2}}}{d_{IJ}^2} \int_{I^c} H_3 h_I,$$

which immediately implies (3.3). In the last estimate we used the following facts,
(i) $\int \frac{dx}{\rho_{xI}^s} \sim \frac{1}{|I|^{s-1}}$ for any $1 < s$;
(ii) $\int_{I^c} \frac{dx}{\rho_{xJ}^2} \sim 1/d_{IJ}$ for $J \subset I$.

This type of integrals/series will appear repeatedly. Variations will be introduced as subtler sets/intervals are defined.

Theorem 3.1. *Suppose (u, v) satisfy conditions (a), (b), and (c). Then the Hilbert transform H is bounded from $L^2(u)$ to $L^2(v)$.*

Proof. Our goal is to show that the operator $v^{\frac{1}{2}} H u^{-\frac{1}{2}}$ is bounded on $L^2(\mathbb{R}) \otimes \mathcal{H}$. By condition (a) which states that operators $v^{\frac{1}{2}} M_v^{-\frac{1}{2}}$ and $u^{-\frac{1}{2}} M_u^{-\frac{1}{2}}$ are bounded, it would suffice to show that $M_v^{\frac{1}{2}} H M_u^{\frac{1}{2}}$ is bounded. In fact, we will show that $M_v^{\frac{1}{2}}(H_1+H_2+H_5)M_u^{\frac{1}{2}}$, as well as $v^{\frac{1}{2}} H_3 M_u^{\frac{1}{2}}$ and $M_v^{\frac{1}{2}} H_4 u^{-\frac{1}{2}}$ are bounded. Then we shall write

$$v^{\frac{1}{2}} H u^{-\frac{1}{2}} = T_v M_v^{\frac{1}{2}}(H_1 + H_2 + H_5) M_u^{\frac{1}{2}} T_u^* + (v^{\frac{1}{2}} H_3 M_u^{\frac{1}{2}}) T_u^* + T_v (M_v^{\frac{1}{2}} H_4 u^{-\frac{1}{2}}),$$

thereby proving the theorem.

By the symmetry between u and v, proving that $M_v^{\frac{1}{2}} H_1 M_u^{\frac{1}{2}}$ and $v^{\frac{1}{2}} H_3 M_u^{\frac{1}{2}}$ are bounded is the same as proving that $M_v^{\frac{1}{2}} H_2 M_u^{\frac{1}{2}}$ and $M_v^{\frac{1}{2}} H_4 u^{-\frac{1}{2}}$ are bounded. The proof bounding $M_v^{\frac{1}{2}} H_5 M_u^{\frac{1}{2}}$ is also exactly the same as the proof that $M_v^{\frac{1}{2}} H_1 M_u^{\frac{1}{2}}$ is bounded once one makes the trivial observation that for *any* two intervals I and J *of the same length*, one has

$$|\langle Hh_I, h_J \rangle| \leq C \frac{|I|^{\frac{3}{2}} |J|^{\frac{3}{2}}}{\rho_{IJ}^3},$$

where ρ_{IJ} is the maximum of $|I|$, $|J|$, and the distance between I and J.

Thus we shall only prove that $M_v^{\frac{1}{2}} H_1 M_u^{\frac{1}{2}}$ and $v^{\frac{1}{2}} H_3 M_u^{\frac{1}{2}}$ are bounded.

We begin with $M_v^{\frac{1}{2}} H_1 M_u^{\frac{1}{2}}$. We shall denote its matrix coefficients by K_{IJ}. Each is a linear operator on $\mathcal{H}$. We have that for $|I| > |J|$ with $3I \cap 3J = \emptyset$,

$$K_{IJ} = v_I^{\frac{1}{2}} H_{IJ} u_J^{\frac{1}{2}}.$$

Here H_{IJ} denotes the real, scalar coefficient of the Hilbert transform. For these (I, J)'s, one has the classical estimate see [Da,TV],

$$|H_{IJ}| \leq C \frac{|I|^{\frac{3}{2}} |J|^{\frac{3}{2}}}{\rho_{IJ}^3}.$$

Throughout this section whenever A is a real scalar or more generally a self-adjoint operator, we shall, by abuse of notation denote by $A^{\frac{1}{2}}$ some choice of normal square root for A always using the fact that $A^{\frac{1}{2}}(A^{\frac{1}{2}})^* = |A|$ where $|A|$ denotes the sum of the positive and negative parts of A.

We apply Lemma 2.1 to K_{IJ}. We let $A_{IJ} = u_I^{\frac{1}{2}} v_J^{\frac{1}{2}} H_{IJ}^{\frac{1}{2}}$. Hence, we let $B_{IJ} = H_{IJ}^{\frac{1}{2}}$. The desired estimate on $\sum_I B_{IJ}^* B_{IJ}$ is simply the corresponding estimate for the scalar Hilbert transform. We need only bound

$$\sum_{J:|J|<|I|;3J\cap 3I=\emptyset} u_I^{\frac{1}{2}} v_J |H_{IJ}| u_I^{\frac{1}{2}} \leq C u_I^{\frac{1}{2}} \Bigg(\sum_{J:|J|<|I|;3J\cap 3I=\emptyset} \frac{v_J |J|^{\frac{3}{2}} |I|^{\frac{3}{2}}}{\rho_{IJ}^3} \Bigg) u_I^{\frac{1}{2}},$$

where the last inequality is in the sense of positive operators. Suppose that $I \in \mathcal{D}_j$. Then we subdivide into a sum over the intervals $J \in \mathcal{D}_k$ and over all $k > j$. Hence,

$$\sum_{J\in\mathcal{D}_k} A_{IJ} A_{IJ}^* \leq C \sum_{k>j} u_I^{\frac{1}{2}} \Bigg(\sum_{J:J\in\mathcal{D}_k;3J\cap 3I=\emptyset} \frac{v_J |J|^{\frac{3}{2}} |I|^{\frac{3}{2}}}{\rho_{IJ}^3} \Bigg) u_I^{\frac{1}{2}}. \tag{3.8}$$

Now we estimate
(3.9)
$$\sum_{J:J\in\mathcal{D}_k;3J\cap 3I=\emptyset}\frac{v_J|J|^{\frac{3}{2}}|I|^{\frac{3}{2}}}{\rho_{IJ}^3}$$
$$\leq\Big(\sum_{J\in\mathcal{D}_k}\frac{|I|^{\frac{3}{2}}|J|^{\frac{3}{2}}}{\rho_{IJ}^3}\Big)^{\frac{\epsilon}{1+\epsilon}}\Big(\sum_{J\in\mathcal{D}_k;3J\cap 3I=\emptyset}\frac{|I|^{\frac{3}{2}}|J|^{\frac{3}{2}}v_J^{1+\epsilon}}{\rho_{IJ}^3}\Big)^{\frac{1}{1+\epsilon}}$$
$$\leq C\Big(\frac{|J|^{\frac{1}{2}}}{|I|^{\frac{1}{2}}}\Big)^{\frac{\epsilon}{1+\epsilon}}\Big(\sum_{J\in\mathcal{D}_k;3J\cap 3I=\emptyset}\frac{|J|^{\frac{3}{2}}|I|^{\frac{1}{2}}v_J^{1+\epsilon}}{\rho_{IJ}^2}\Big)^{\frac{1}{1+\epsilon}}$$
$$\leq C\frac{|J|^{\frac{1}{2}}}{|I|^{\frac{1}{2}}}\Big(|I|\int_{(3I)^c}\frac{v^{1+\epsilon}}{(x-y_I)^2}\Big)^{\frac{1}{1+\epsilon}}$$

Now plugging (3.9) into (3.8) and using condition (b) we conclude

$$\sum_J A_{IJ}A_{IJ}^*\leq\sum_{k>j}\frac{|J|^{\frac{1}{2}}}{|I|^{\frac{1}{2}}}.$$

But this is a geometric sum, so that $M_v^{\frac{1}{2}}H_1M_u^{\frac{1}{2}}$ is bounded.

For the penultimate inequality in (3.9) we used the fact that:
(iii) $\sum_{J\in D_k}\frac{1}{\rho_{IJ}^s}\sim\frac{|I|}{|J||I|^s}$ for $1<s$, $|J|<|I|$;
which we can compare to
(iv) $\sum_{I\in D_j}\frac{1}{\rho_{IJ}^s}\sim\frac{1}{|I|^s}$ for $1<s$, $|J|<|I|$.
(v) $\sum_{I\in D_j}\frac{|I|}{\rho_{xI}^s}\sim\frac{|I|}{|I|^s}$ for $1<s$.

Now, we write $L=v^{\frac{1}{2}}(H_3-\pi_{c^v}^*)M_u^{-\frac{1}{2}}$. If we can bound the operator L then we have proven the theorem by condition (c). We shall apply Cotlar's lemma writing $L=\sum_j L_j$ with $L_j=L\Delta_j$, where $\Delta_j=\sum_{I\in\mathcal{D}_j}E_I$. We must bound $L_k^*L_j$, and it will be enough to consider only $k\leq j$ by symmetry considerations the case $k\geq j$ can be done using the half of hypothesis (a), (b), (c) that are not used in what follows. Also we should not worry about bounding $L_kl_j^*$ because it can be seen that for the nonselfadjoint part of L it will give zero.

We write with $I\in\mathcal{D}_j$ and with $J\in\mathcal{D}_k$ and with $z_1,z_2\in\mathcal{H}$ that

$$\langle L(h_I\otimes z_1),L(h_J\otimes z_2)\rangle_{L^2(\mathbb{R},\mathcal{H})}=\sum_{\alpha=1}^{6}\langle z_1,L_{IJ}^{\alpha}z_2\rangle_{\mathcal{H}}.$$

To define the decomposition L^α we now define a set of intervals B_{jk}. An interval J is contained in B_{jk} precisely if $J \in \mathcal{D}_k$ and there exists a $I \in \mathcal{D}_j$ so that the distance between J and ∂I is bounded by $|I|^{\frac{1}{2}}|J|^{\frac{1}{2}}$. Now we define:

$$\langle z_1, L^1_{IJ} z_2\rangle_{\mathcal{H}} = \langle v^{\frac{1}{2}} H_3 h_I u_I^{\frac{1}{2}} z_1, v^{\frac{1}{2}} H_3 h_J u_J^{\frac{1}{2}} z_2\rangle_{L^2(\mathbb{R},\mathcal{H})},$$

when $J \in B_{jk}$ and 0 otherwise,

$$-\langle z_1, L^2_{IJ} z_2\rangle_{\mathcal{H}} = \langle v^{\frac{1}{2}} H_3 h_I u_I^{\frac{1}{2}} z_1, v^{\frac{1}{2}} \pi^*_{c^v} M_u^{\frac{1}{2}}(h_J \otimes z_2)\rangle_{L^2(\mathbb{R},\mathcal{H})}$$

when $J \in B_{jk}$ and 0 otherwise,

$$-\langle z_1, L^3_{IJ} z_2\rangle_{\mathcal{H}} = \langle v^{\frac{1}{2}} \pi^*_{c^v} M_u^{\frac{1}{2}}(h_I \otimes z_1), v^{\frac{1}{2}} H_3 h_J u_J^{\frac{1}{2}} z_2\rangle_{L^2(\mathbb{R},\mathcal{H})}$$

when $J \in B_{jk}$ and 0 otherwise,

$$\langle z_1, L^4_{IJ} z_2\rangle_{\mathcal{H}} = \langle v^{\frac{1}{2}} \pi^*_{c^v} M_u^{\frac{1}{2}}(h_I \otimes z_1), v^{\frac{1}{2}} \pi^*_{c^v} M_u^{\frac{1}{2}}(h_J \otimes z_2)\rangle_{L^2(\mathbb{R},\mathcal{H})}$$

when $J \in B_{jk}$ and 0 otherwise,

$$\langle z_1, L^5_{IJ} z_2\rangle_{\mathcal{H}} = \langle v^{\frac{1}{2}} H_3 h_I u_I^{\frac{1}{2}} z_1, L(h_J \otimes z_2)\rangle_{L^2(\mathbb{R},\mathcal{H})}$$

when $J \notin B_{jk}$ and 0 otherwise and

$$-\langle z_1, L^6_{IJ} z_2\rangle_{\mathcal{H}} = \langle v^{\frac{1}{2}} \pi^*_{c^v} M_u^{\frac{1}{2}}(h_I \otimes z_1), L(h_J \otimes z_2)\rangle_{L^2(\mathbb{R},\mathcal{H})}$$

when $J \notin B_{jk}$ and 0 otherwise.

It suffices to bound the operator-valued matrices L^α with exponential decay in $|j-k|$, and this is what we shall do.

The main point of the argument is as follows. By the definition of H_3, the Haar expansion of $H_3 h_I$ is the sum of all components $\langle H h_I, h_J\rangle h_J$ for J such that $3I \cap 3J \neq \emptyset$ and $|I| < |J|$, denote that collection of intervals by $Z_3(I)$, for each I. The dyadic intervals J with respect to the property $3I \cap 3J = \emptyset$ which have $|I| \geq |J|$ and such that their parents $\tilde{J}$ belongs to $Z_3(I)$ form a disjoint covering of $(5I)^c$ and we may define $J_I(x)$ for any point x in $(5I)^c$ to be the element of this covering containing x. For $x \in 5I$ we define $J_I(x)$ to be the dyadic interval of length $|I|$ containing x.

Then, by definition, we have that

$$(H_3 h_I)(x) = m_{J_I(x)} H h_I.$$

Thus while Hh_I has logarithmic singularities, the function H_3h_I does not. In fact, we have the precise size estimates which we shall use from now on:

$$|H_3h_I(x)| \leq C\frac{|I|^{\frac{3}{2}}}{\rho_{xI}^2}. \tag{3.10}$$

Let us state some facts that will be used often in the proof. For $I \in \mathcal{D}_j$, $J \in \mathcal{D}_k$, and $j < k$,
(vi) $\sum_{J\in B_{jk}} \frac{1}{\rho_{xJ}^2} \leq \frac{C}{|J|^2}$ for $x \in S_{jk}$;
(vii) $\sum_{J\in B_{jk}} \frac{1}{\rho_{xJ}^2} \leq \frac{C}{|J||I|}$ for $x \in S_{jk}^c$;
(viii) $\int_{S_{jk}} \frac{1}{\rho_{xI}^2} \leq C\frac{|J|^{1/2}}{|I|^{3/2}}$.

We begin with $L_{IJ}^1 = u_I^{\frac{1}{2}}\big(\int v(H_3h_I)(H_3h_J)\big)u_J^{\frac{1}{2}}$. We apply Lemma 2.1, letting $A_{IJ} = u_I^{\frac{1}{2}}\big(\int v(H_3h_I)(H_3h_J)\big)^{\frac{1}{2}}$, and $B_{IJ} = \big(\int v(H_3h_I)(H_3h_J)\big)^{\frac{1}{2}}u_J^{\frac{1}{2}}$. We let $S_{jk} = 2(\cup_{J\in B_{jk}} J)$ and shall estimate separately the integral on S_{jk} and on S_{jk}^c. We must estimate

$$\begin{aligned}\sum_{J\in B_{jk}} A_{IJ}A_{IJ}^* &= \sum_{J\in B_{jk}} u_I^{\frac{1}{2}}\Big|\int v(H_3h_I)(H_3h_J)\Big|u_I^{\frac{1}{2}} \\ &\leq \sum_{J\in B_{jk}} u_I^{\frac{1}{2}}\bigg(\int_S \frac{v|I|^{\frac{3}{2}}|J|^{\frac{3}{2}}}{\rho_{xI}^2\rho_{xJ}^2} + \int_{S^c} \frac{v|I|^{\frac{3}{2}}|J|^{\frac{3}{2}}}{\rho_{xI}^2\rho_{xJ}^2}\bigg)u_I^{\frac{1}{2}}.\end{aligned} \tag{3.11}$$

Here again $|.|$ denotes the sum of the positive and negative parts. Now, we estimate the integrals using the trivial bound $|J| \leq \rho_{xJ}$, Holder, (vi) and (viii),
(3.12)

$$\begin{aligned}&\sum_{J\in B_{jk}} \int_{S_{jk}} \frac{v|I|^{\frac{3}{2}}|J|^{\frac{3}{2}}}{\rho_{xI}^2\rho_{xJ}^2} = \int_{S_{jk}} \frac{v|I|^{\frac{3}{2}}}{\rho_{xI}^2}\bigg(\sum_{J\in B_{jk}} \frac{|J|^{\frac{3}{2}}}{\rho_{xJ}^2}\bigg) \\ &\leq \Big(\frac{|I|}{|J|}\Big)^{\frac{1}{2}}\bigg(|I|\int_{S_{jk}} \frac{v}{\rho_{xI}^2}\bigg) \leq C\Big(\frac{|I|}{|J|}\Big)^{\frac{1}{2}}\bigg(|I|\int_{S_{jk}} \frac{v^{1+\epsilon}}{\rho_{xI}^2}\bigg)^{\frac{1}{1+\epsilon}}\bigg(|I|\int_{S_{jk}} \frac{1}{\rho_{xI}^2}\bigg)^{\frac{\epsilon}{1+\epsilon}} \\ &\leq C\Big(\frac{|I|}{|J|}\Big)^{\frac{1}{2}}\Big(\frac{|S_{jk}\cap I|}{|I|}\Big)^{\frac{\epsilon}{1+\epsilon}}\bigg(|I|\int \frac{v^{1+\epsilon}}{\rho_{xI}^2}\bigg)^{\frac{1}{1+\epsilon}} \leq C\Big(\frac{|I|}{|J|}\Big)^{\frac{1}{2}-\frac{\epsilon}{2+2\epsilon}}\bigg(|I|\int \frac{v^{1+\epsilon}}{\rho_{xI}^2}\bigg)^{\frac{1}{1+\epsilon}}\end{aligned}$$

Here the factor of $\big(\frac{|S_{jk}\cap I|}{|I|}\big)$ comes from the fact that ρ_{xI} is almost constant on intervals of length I, also remember that $|S_{jk}\cap I| \sim (|I||J|)^{\frac{1}{2}}$. However for $x \in S^c$ we may use that

$\rho_{xJ} \geq (|I||J|)^{\frac{1}{2}}$, thus obtaining, using (vii), and Holder,

$$\begin{aligned}
&\sum_{J\in B_{jk}} \int_{S_{jk}^c} \frac{v|I|^{\frac{3}{2}}|J|^{\frac{3}{2}}}{\rho_{xI}^2\rho_{xJ}^2} \leq \Big(\frac{|J|}{|I|}\Big)^{\frac{1}{2}}\Big(|I|\int_{S_{jk}^c}\frac{v}{\rho_{xI}^2}\Big)\\
&\leq C\Big(\frac{|J|}{|I|}\Big)^{\frac{1}{2}}\Big(|I|\int\frac{v}{\rho_{xI}^2}\Big) \leq C\Big(\frac{|J|}{|I|}\Big)^{\frac{1}{2}}\Big(|I|\int\frac{1}{\rho_{xI}^2}\Big)^{\frac{\epsilon}{1+\epsilon}}\Big(|I|\int\frac{v^{1+\epsilon}}{\rho_{xI}^2}\Big)^{\frac{1}{1+\epsilon}} \qquad (3.13)\\
&\leq C\Big(\frac{|J|}{|I|}\Big)^{\frac{1}{2}}(|I|\int\frac{v^{1+\epsilon}}{\rho_{xI}^2})^{\frac{1}{1+\epsilon}}
\end{aligned}$$

Now we plug (3.12) and (3.13) into (3.11) to obtain

$$\sum_{J\in B_{jk}} A_{IJ}A_{IJ}^* \leq C\Big[\frac{|J|}{|I|}+\Big(\frac{|I|}{|J|}\Big)^{\frac{1}{2}-\frac{\epsilon}{2+2\epsilon}}\Big].$$

We compute directly

$$\sum_I B_{IJ}^*B_{IJ} \leq \sum_I u_J^{\frac{1}{2}}(\int\frac{v|I|^{\frac{3}{2}}|J|^{\frac{3}{2}}}{\rho_{xI}^2\rho_{xJ}^2})u_J^{\frac{1}{2}} \leq C\frac{|J|^{\frac{1}{2}}}{|I|^{\frac{1}{2}}}.$$

Here the last inequality follows from summing inside the integral, and then applying Hölder, condition (b), (i) and (v) as in (3.12) and (3.13). But this provides the desired estimates on L_{IJ}^1 since the product of the estimates on $\sum A_{IJ}A_{IJ}^*$ and on $\sum B_{IJ}^*B_{IJ}$ decays exponentially in $k-j$.

We will use repeatedly the estimate deduced by Hölder and (b) in (3.13)

$$u_I^{1/2}\Big(\int\frac{|I|v(x)}{\rho_{xI}^2}\Big)u_I^{1/2} \leq C$$

To bound L_{IJ}^2, naturally, we shall use condition (c), recalling the estimate (3.5) namely

$$\|u_I^{\frac{1}{2}}c_I^v(m_Iv)^{\frac{1}{2}}\|_{\mathcal{H}\to\mathcal{H}} \leq C\sqrt{|I|}. \qquad (3.5)$$

We abbreviate $D_I = u_I^{\frac{1}{2}}c_I^v(m_Iv)^{\frac{1}{2}}$. By definition, we have that for $J\in B_{jk}$, and $x_J\in J$,

$$L_{IJ}^2 = u_I^{\frac{1}{2}}(H_3h_I)(x_J)(m_Jv)^{\frac{1}{2}}D_J^*,$$

remember that

$$\pi_{c^v}^*M_u^{1/2}(h_J\otimes z) = \frac{\chi_J(x)}{|J|}(c_J^v)^*u_J^{1/2}z.$$

We apply Lemma 3.1, letting

$$A_{IJ} = u_I^{\frac{1}{2}}(m_J v)^{\frac{1}{2}}\frac{|I|^{\frac{1}{2}}|J|^{\frac{1}{2}}}{\rho_{IJ}}$$

and letting

$$B_{IJ} = \frac{\rho_{IJ}}{|I|^{\frac{1}{2}}|J|^{\frac{1}{2}}}(H_3 h_I)(x_J)D_J^*.$$

Now we compute

$$\sum_{J\in B_{jk}} A_{IJ}A_{IJ}^* = u_I^{\frac{1}{2}}|I|\Big(\sum_{J\in B_{jk}}\frac{|J|m_J v}{\rho_{IJ}^2}\Big)u_I^{\frac{1}{2}}$$
$$\leq Cu_I^{\frac{1}{2}}\left(|I|\int_{S_{jk}}\frac{v}{\rho_{xI}^2}\right)u_I^{\frac{1}{2}}$$
$$\leq C\Big(\frac{|J|}{|I|}\Big)^{\frac{\epsilon}{2+2\epsilon}}u_I^{\frac{1}{2}}\left(|I|\int\frac{v^{1+\epsilon}}{\rho_{xI}^2}\right)^{\frac{1}{1+\epsilon}}u_I^{\frac{1}{2}} \leq C\Big(\frac{|J|}{|I|}\Big)^{\frac{\epsilon}{2+2\epsilon}}$$

Here S_{jk} is as in (3.12), the penultimate inequality comes from the same application of Hölder as in (3.12), and the final inequality comes from condition (b). On the other hand, by (3.5), we see that

$$\sum_I B_{IJ}^* B_{IJ} \leq \sum_I \frac{|I|^2}{\rho_{IJ}^2} \leq C.$$

Hence, we have obtained the desired estimate for L_{IJ}^2.

Next, we estimate L_{IJ}^3. We obtain immediately from the definition that

$$-L_{IJ}^3 = \frac{1}{|I|}u_I^{\frac{1}{2}}c_I^v\left(\int_I v(H_3 h_J)\right)u_J^{\frac{1}{2}}.$$

As before, we shall use that from condition (c), we have (3.5). We apply Lemma 3.1, letting

$$A_{IJ} = \frac{|J|^{\frac{1}{4}}}{|I|}u_I^{\frac{1}{2}}c_I^v\Big(\int_I vH_3h_J\Big)^{\frac{1}{2}}, \tag{3.14}$$

and letting

$$B_{IJ} = |J|^{-\frac{1}{4}}\left(\int_I vH_3h_J\right)^{\frac{1}{2}}u_J^{\frac{1}{2}}.$$

We compute

$$\sum_{J\in B_{jk}} A_{IJ}A^*_{IJ} \le \sum_{J\in B_{jk}} \frac{1}{|I|^2} u_I^{\frac{1}{2}} c_I^v \Big(\int_I |J|^2 \frac{v}{\rho^2_{xJ}} \Big) (c_I^v)^* u_I^{\frac{1}{2}}, \tag{3.15}$$

by plugging in (3.14) into the sum and using the size estimates on $H_3 h_J$. Now, we estimate the integral in (3.15) by breaking up the interval I into $I \cap S_{jk}$ and $I \cap S^c_{jk}$ observing, by summing under the integral and using (vi) and (vii), that

$$\sum_{J\in B_{jk}} \int_I |J|^2 \frac{v}{\rho^2_{xJ}} \le C \Big(\int_{S_{jk}\cap I} v + \frac{|J|^{\frac{1}{2}}}{|I|^{\frac{1}{2}}} \int_I v \Big). \tag{3.16}$$

The second piece in (3.16) is clearly bounded by $|J|^{\frac{1}{2}}|I|^{\frac{1}{2}} m_I v$. As for the first piece, we use doubling observing that $S\cap I$ is contained in the rightmost and leftmost dyadic subintervals of I having measure more that $2|J|^{\frac{1}{2}}|I|^{\frac{1}{2}}$. Recall doubling implies that if K is any dyadic interval and $\tilde{K}$ its parent then

$$\int_{\tilde{K}} v \le C \int_K v.$$

Now let K_b be K's twin sister. Since $\int_K v \ge \frac{1}{C} \int_{\tilde{K}} v$ while $\int_{\tilde{K}} v = \int_K v + \int_{K_b} v$, one has that

$$\int_{K_b} v \le \big(1 - \frac{1}{C}\big) \int_{\tilde{K}} v.$$

Naturally, the same holds for K by applying the doubling condition on K_b. In fact if K' is any descendant of K after l generations, one has

$$\int_{K'} v \le \big(1 - \frac{1}{C}\big)^l \int_K v \le \big(\frac{|K'|}{|K|}\big)^{\delta} \int_K v,$$

where $\delta \ge 0$ depends only on the doubling constant. Now since $S\cap I$ is contained in two descendants of I of length at most $4|J|^{\frac{1}{2}}|I|^{\frac{1}{2}}$, one has that

$$\int_{S\cap I} v \le C \big(\frac{|J|}{|I|}\big)^{\frac{\delta}{2}} \int_I v. \tag{3.17}$$

Plugging our observations into (3.16) yields that

$$\sum_{J\in B_{jk}} \int_I |J|^2 \frac{v}{\rho^2_{xJ}} \le C \big(|I|^{1-\frac{\delta}{2}} |J|^{\frac{\delta}{2}} + |I|^{\frac{1}{2}} |J|^{\frac{1}{2}} \big) m_I v. \tag{3.18}$$

Now we plug (3.18) into (3.15), applying (3.5) and the fact that when $P_1 \leq P_2$ then $TP_1T^* \leq TP_2T^*$ for any P_1, P_2, and T to obtain that

$$\sum_{J \in B_{jk}} A_{IJ}A_{IJ}^* \leq C\Big((\frac{|J|}{|I|})^{\frac{\delta}{2}} + (\frac{|J|}{|I|})^{\frac{1}{2}} \Big).$$

Now,

$$\sum_{I \in \mathcal{D}_j} B_{IJ}^* B_{IJ} \leq \sum_{I \in \mathcal{D}_j} u_J^{\frac{1}{2}} \Big(\int_I \frac{|J|v}{\rho_{xJ}^2} \Big) u_J^{\frac{1}{2}} \leq u_J^{\frac{1}{2}} \Big(\int \frac{|J|v}{\rho_{xJ}^2} \Big) u_J^{\frac{1}{2}} \leq C u_J^{\frac{1}{2}} v_J u_J^{\frac{1}{2}} \leq C.$$

Here the penultimate inequality is obtained by using Hölder and condition (b) as in (3.12) and (3.13). Thus, we have obtained the desired estimates on L_{IJ}^3.

We come now to L_{IJ}^4. By definition, when $J \in B_{jk}$ and $J \subset I$,

$$L_{IJ}^4 = u_I^{\frac{1}{2}} c_I^v \Big(\int_J \frac{v}{|I||J|} \Big) (c_J^v)^* u_J^{\frac{1}{2}} = \Big(\frac{1}{|I|} \Big) D_I (m_I v)^{-\frac{1}{2}} (m_J v)^{\frac{1}{2}} D_J^*;$$

when $J \cap I = \emptyset$ then by support considerations $L_{IJ}^4 = 0$.

As usual we apply Lemma 2.1, though the sum over I will be over a set with only one element. We let

$$A_{IJ} = \frac{|J|^{\frac{1}{2}}}{|I|} D_I (m_I v)^{-\frac{1}{2}} (m_J v)^{\frac{1}{2}},$$

and

$$B_{IJ} = \frac{1}{|J|^{\frac{1}{2}}} D_J^*.$$

We compute

$$\begin{aligned} \sum_{J \in B_{jk}; J \subset I} A_{IJ} A_{IJ}^* &= \sum_{J \in B_{jk}; J \subset I} \Big(\frac{1}{|I|} \Big) D_I \Big(\frac{|J|}{|I|} \Big) (m_I v)^{-\frac{1}{2}} (m_J v)(m_I v)^{-\frac{1}{2}} D_I^* \\ &\leq \Big(\frac{1}{|I|} \Big)^2 D_I (m_I v)^{-\frac{1}{2}} \int_{S \cap I} v (m_I v)^{-\frac{1}{2}} D_I^* \\ &\leq C \Big(\frac{1}{|I|} \Big) D_I \Big(\frac{|J|}{|I|} \Big)^{\frac{\delta}{2}} D_I^* \leq C \Big(\frac{|J|}{|I|} \Big)^{\delta}. \end{aligned}$$

Here the penultimate estimate is by (3.17) and the last one by (3.5). The bound on $B_{IJ}^* B_{IJ}$ independent of J is just (3.5). Hence, L_{IJ}^4 satisfies the desired estimates.

Next we bound L^5_{IJ}. This time $J \notin B_{jk}$. We define $\hat{J} = (\frac{|I|^{\frac{1}{2}}}{|J|^{\frac{1}{2}}})J$, and we have that if $J \subset I$ then $\hat{J} \subset I$. Now the reason that we are Cotlarizing $L = v^{\frac{1}{2}}(H_3 - \pi^*_{c^v})M_u^{\frac{1}{2}}$ instead of just $v^{\frac{1}{2}}H_3 M_u^{\frac{1}{2}}$ is precisely that it gives us the cancelation

$$\int v^{\frac{1}{2}} L h_I = 0,$$

for every interval I. We now simply use the fact that $H_3 h_I$ is constant on $\hat{J}$ while $\int_{\hat{J}} v^{\frac{1}{2}} L h_J = -\int_{\hat{J}^c} v^{\frac{1}{2}} L h_J$ to write

$$L^5_{IJ} = L^{5,1}_{IJ} - L^{5,2}_{IJ} = u_I^{\frac{1}{2}} \big(\int_{(\hat{J})^c} v(H_3 h_I)(H_3 h_J) \big) u_J^{\frac{1}{2}} - u_I^{\frac{1}{2}} (H_3 h_I)(x_J) \left(\int_{(\hat{J})^c} v(H_3 h_J) \right) u_J^{\frac{1}{2}}.$$

First, we bound $L^{5,1}_{IJ}$ by Lemma 2.1. We let

$$A_{IJ} = u_I^{\frac{1}{2}} \big(\int_{(\hat{J})^c} v(H_3 h_I)(H_3 h_J) \big)^{\frac{1}{2}},$$

and we let

$$B_{IJ} = \big(\int_{(\hat{J})^c} v(H_3 h_I)(H_3 h_J) \big)^{\frac{1}{2}} u_J^{\frac{1}{2}}.$$

We have

$$\sum_{J \notin B_{jk}} A_{IJ} A^*_{IJ} \le u_I^{\frac{1}{2}} \big(\sum_{J \notin B_{jk}} \int_{(\hat{J})^c} \frac{v |I|^{\frac{3}{2}} |J|^{\frac{3}{2}}}{\rho^2_{xI} \rho^2_{xJ}} \big) u_I^{\frac{1}{2}} \le C u_I^{\frac{1}{2}} \big(\int \frac{v|I|}{\rho^2_{xI}} \big) u_I^{\frac{1}{2}} \le C.$$

Here, the penultimate inequality comes from the simple observation that for each x,

$$\sum_{J \in \mathcal{D}_k} \frac{1}{\rho^2_{xJ}} \chi_{\hat{J}^c}(x) \le \frac{C}{|J|^{\frac{3}{2}} |I|^{\frac{1}{2}}} \tag{3.19}$$

The inequality (3.19) is obtained by majorizing the sum by $\frac{1}{|J|} \int_{|J|^{\frac{1}{2}} |I|^{\frac{1}{2}}}^{\infty} \frac{dx}{x^2}$. Furthermore,

$$\begin{aligned} \sum_{I \in \mathcal{D}_j} B^*_{IJ} B_{IJ} &\le u_J^{\frac{1}{2}} \left(\sum_{I \in \mathcal{D}_j} \int_{(\hat{J})^c} \frac{v |I|^{\frac{3}{2}} |J|^{\frac{3}{2}}}{\rho^2_{xI} \rho^2_{xJ}} \right) u_J^{\frac{1}{2}} \\ &\le C \left(\frac{|J|^{\frac{1}{2}}}{|I|^{\frac{1}{2}}} \right) \left(u_J^{\frac{1}{2}} \int \frac{|J| v}{\rho^2_{xJ}} \right) u_J^{\frac{1}{2}} \le C \left(\frac{|J|}{|I|} \right)^{\frac{1}{2}}, \end{aligned}$$

which is the desired estimate for $L^{5,1}_{IJ}$. Redefining A_{IJ} and B_{IJ}, we continue by decomposing $L^{5,2}_{IJ} = A_{IJ}B_{IJ}$. First, we let $K_{IJ} = \rho_{IJ}^{\frac{1}{1+\epsilon}}(H_3h_I)(J)$, and we let

$$A_{IJ} = |I|^{\frac{1}{2(1+\epsilon)}} u_I^{\frac{1}{2}} \rho_{IJ}^{\frac{-1}{1+\epsilon}} K_{IJ}^{\frac{1}{2}} \big(\int_{(\hat{J})^c} v(H_3h_J)\big)^{\frac{1}{2}},$$

and

$$B_{IJ} = (|I|)^{-\frac{1}{2(1+\epsilon}} K_{IJ}^{\frac{1}{2}} \big(\int_{(\hat{J})^c} v(H_3h_J)\big)^{\frac{1}{2}} u_J^{\frac{1}{2}}.$$

We estimate

$$\begin{aligned} A_{IJ}A^*_{IJ} &\leq u_I^{\frac{1}{2}} |K_{IJ}||J|^{\frac{3}{2}} \left(|I| \int_{\hat{J}^c} \frac{v^{1+\epsilon}}{\rho^2_{xJ}\rho^2_{IJ}} \right)^{\frac{1}{1+\epsilon}} \left(\int_{\hat{J}^c} \frac{1}{\rho^2_{xJ}} \right)^{\frac{\epsilon}{1+\epsilon}} u_I^{\frac{1}{2}} \\ &\leq C|K_{IJ}\|J|^{\frac{3}{2}} u_I^{\frac{1}{2}} \left(|I| \int_{\hat{J}^c} \frac{v^{1+\epsilon}}{\rho^2_{xI}} \right)^{\frac{1}{1+\epsilon}} u_I^{\frac{1}{2}} \left(\frac{1}{|J|^{\frac{1}{2}}|I|^{\frac{1}{2}}} \right)^{\frac{\epsilon}{1+\epsilon}} (|I||J|)^{-\frac{1}{1+\epsilon}} \\ &\leq C \frac{(|I||J|)^{\frac{1+2\epsilon}{2(1+\epsilon)}}}{\rho_{IJ}^{\frac{1+2\epsilon}{1+\epsilon}}}. \end{aligned}$$

Here we have used the fact that on $\hat{J}^c$, one has $\rho^2_{IJ}\rho^2_{xJ} \geq |I||J|\rho^2_{xI}$. We sum obtaining

$$\sum_{J\in\mathcal{D}_k} A_{IJ}A^*_{IJ} \leq \sum_{J\in\mathcal{D}_k} \frac{|I|^{\frac{1+2\epsilon}{2(1+\epsilon)}}|J|^{\frac{1+2\epsilon}{2(1+\epsilon)}}}{\rho_{IJ}^{\frac{1+2\epsilon}{1+\epsilon}}} \leq \left(\frac{|I|}{|J|} \right)^{\frac{1}{2(1+\epsilon)?}}.$$

Meanwhile, we compute

$$\begin{aligned} B^*_{IJ}B_{IJ} &\leq (|I||J|)^{-\frac{2+\epsilon}{2(1+\epsilon)}} |J|^{\frac{3}{2}} |K_{IJ}| u_J^{\frac{1}{2}} \left(|J| \int_{\hat{J}^c} \frac{v^{1+\epsilon}}{\rho^2_{xJ}} \right)^{\frac{1}{1+\epsilon}} u_J^{\frac{1}{2}} \\ &\leq C \frac{(|I||J|)^{\frac{1+2\epsilon}{2(1+\epsilon)}}}{\rho_{IJ}^{\frac{1+2\epsilon}{1+\epsilon}}}. \end{aligned}$$

We conclude that

$$\sum_{I\in\mathcal{D}_j} B^*_{IJ}B_{IJ} \leq \sum_{I\in\mathcal{D}_j} \frac{|I|^{\frac{1+2\epsilon}{2(1+\epsilon)}}|J|^{\frac{1+2\epsilon}{2(1+\epsilon)}}}{\rho_{IJ}^{\frac{1+2\epsilon}{1+\epsilon}}} \leq \left(\frac{|J|}{|I|} \right)^{\frac{1+2\epsilon}{2(1+\epsilon)}}.$$

which gives the desired estimate on $L^{5,2}_{IJ}$.

Finally, we come to L^6_{IJ}. We break up into $L^6_{IJ} = L^{6,1}_{IJ} + L^{6,2}_{IJ}$. Here, we let $L^{6,1}_{IJ} = L^6_{IJ}$ when $J \subset I$ and 0 otherwise. As before, we let $D_I = u_I^{\frac{1}{2}} c_I^v (m_I v)^{\frac{1}{2}}$. We let

$$F_{IJ} = (m_I v)^{-\frac{1}{2}} \left(\frac{1}{|I|} \int_{I^c} v H_3 h_J \right) u_J \left(\frac{1}{|I|} \int_{I^c} v H_3 h_J \right) (m_I v)^{-\frac{1}{2}}.$$

We have $\|D_I\| \leq C|I|^{\frac{1}{2}}$ and we have

$$F_{IJ} \leq \frac{|J|^3}{d_{IJ}^4} u_I^{-\frac{1}{2}} u_J u_I^{-\frac{1}{2}}.$$

Recall d_{IJ} is the distance from J to the boundary of I. Here we are directly applying (3.3). Notice this is the only place where we use it. Since all J's we are considering are not in B_{jk}, we have that $d_{IJ} \geq \sqrt{|I|}\sqrt{|J|}$, but for most J, it is even bigger. We write by definition and cancellation,

$$-L^{6,1}_{IJ} = u_I^{\frac{1}{2}} c_I^v \left(\frac{1}{|I|} \int_{I^c} v H_3 h_J \right) u_J^{\frac{1}{2}}.$$

We let $A_{IJ} = L^{6,1}_{IJ}$ and $B_{IJ} = 1$. Then we have

$$\sum_{J : J \in \mathcal{D}_k, J \notin B_{jk}} A_{IJ} A_{IJ}^* = \sum_J D_I F_{IJ} D_I^* \leq D_I \Bigg(\sum_{J \in \mathcal{D}_k} \frac{|J|^3}{d_{IJ}^4} u_I^{-\frac{1}{2}} u_J u_I^{-\frac{1}{2}} \Bigg) D_I^* \leq \frac{|J|^\delta}{|I|^{1+\delta}} D_I D_I^* \leq C \left(\frac{|J|}{|I|} \right)^\delta,$$

for some $\delta > 0$ which is the desired estimate on $L^{6,1}_{IJ}$, provided we can show that

$$\sum_{J : J \in \mathcal{D}_k, J \notin B_{jk}} \frac{|J|^3}{d_{IJ}^4} u_J \leq \frac{|J|^\delta}{|I|^{1+\delta}} u_I,$$

where the sum is over J's contained in I with $d_{IJ} \geq |I|^{\frac{1}{2}} |J|^{\frac{1}{2}}$. We let $\mathcal{Z}_1$ be the set of those J's with $d_{IJ} \leq |I|^{\frac{3}{4}} |J|^{\frac{1}{4}}$ and $\mathcal{Z}_2$ be the set of those J's with $d_{IJ} \geq |I|^{\frac{3}{4}} |J|^{\frac{1}{4}}$. We estimate for $\mathcal{Z}_1$, noticing that $\frac{card(\mathcal{Z}_1) \leq \left(|I|\right.}{\left.|J|\right)^{\frac{3}{4}}}$,

$$\begin{aligned}
\sum_{J \in \mathcal{Z}_1} \frac{|J|^3}{d_{IJ}^4} u_J &\leq \sum_{J \in \mathcal{Z}_1} \frac{|J|}{|I|^2} u_J \\
&\leq \left(\sum_{J \in \mathcal{Z}_1} \frac{|J|}{|I|^2} u_J^{1+\epsilon} \right)^{\frac{1}{1+\epsilon}} \left(\sum_{J \in \mathcal{Z}_1} \frac{|J|}{|I|^2} \right)^{\frac{\epsilon}{1+\epsilon}} \\
&\leq \left(\frac{1}{|I|} \right) u_I \left(\frac{|J|}{|I|} \right)^{\frac{\epsilon}{4(1+\epsilon)}},
\end{aligned}$$

while for $\mathcal{Z}_2$,

$$\begin{aligned}\sum_{J\in\mathcal{Z}_2}\frac{|J|^3}{d_{IJ}^4}u_J &\leq \sum_{J\in\mathcal{Z}_2}\frac{|J|^2}{|I|^3}u_J\\ &\leq \left(\frac{|J|}{|I|}\right)\left(\sum_{J\in\mathcal{Z}_2}\frac{|J|}{|I|^2}u_J^{1+\epsilon}\right)^{\frac{1}{1+\epsilon}}\left(\sum_{J\in\mathcal{Z}_2}\frac{|J|}{|I|^2}\right)^{\frac{\epsilon}{1+\epsilon}}\\ &\leq u_I\left(\frac{|J|}{|I|^2}\right).\end{aligned}$$

This leaves us to bound $L_{IJ}^{6,2}$.

By definition, for $I\cap J=\emptyset$,

$$L_{IJ}^{6,2}=D_I(m_Iv)^{-\frac{1}{2}}\left(\frac{1}{|I|}\int_I vH_3h_J\right)u_J^{\frac{1}{2}},$$

zero otherwise.

We break up

$$A_{IJ}=D_I(m_Iv)^{-\frac{1}{2}}\left(\frac{1}{|I|}\int_I vH_3h_J\right)^{\frac{1}{2}},$$

and

$$B_{IJ}=\left(\frac{1}{|I|}\int_I vH_3h_J\right)^{\frac{1}{2}}u_J^{\frac{1}{2}},$$

and now we obtain bounds easily using the fact that $J\cap I=\emptyset$ and $d_{IJ}\geq(|I||J|)^{\frac{1}{2}}$. We simply compute

$$\begin{aligned}\sum_{J:J\in\mathcal{D}_k,J\notin B_{jk}}A_{IJ}A_{IJ}^* &\leq \left(\sup_{x\in I}\sum_{J\in\mathcal{D}_k}\frac{|J|^{\frac{3}{2}}}{\rho_{xJ}^2}\right)D_I(m_Iv)^{-\frac{1}{2}}\left(\frac{1}{|I|}\int_I v\right)(m_Iv)^{-\frac{1}{2}}D_I^*\\ &\leq C\frac{1}{\sqrt{|I|}}D_ID_I^*\leq C\sqrt{|I|}.\end{aligned}$$

At the same time, seeing that the sum on I merely extends the support of the integral, we obtain

$$\sum_I B_{IJ}^*B_{IJ}\leq\frac{|J|^{\frac{1}{2}}}{|I|}u_J^{\frac{1}{2}}\left(|J|\int\frac{v}{\rho_{xJ}^2}\right)u_J^{\frac{1}{2}}\leq C\left(\frac{|J|^{\frac{1}{2}}}{|I|}\right).$$

Multiplying these two estimates and obtaining decay, we prove Theorem 3.1 $\square$

It may be worth pointing out that if assumptions (3.3) and (3.4) seem unappealing, we can also obtain the same result by assuming a sort of "doubling at infinity" condition for v and u^{-1}. Thus it suffices, for example, to assume there is a $\delta > 0$ with

$$\left(\int_{I^c} vH_3h_J\right)(m_Jv)^{-1}\left(\int_{I^c} vH_3h_J\right) \leq \left(\frac{|J|}{d_{IJ}}\right)^{\delta}\left(\int_{J^c} vH_3h_J\right)(m_Jv)^{-1}\left(\int_{J^c} vH_3h_J\right).$$

Then we simply use

$$\|F_{IJ}\| = \|u_J^{\frac{1}{2}}\left(\int_{I^c} vH_3h_J\right)(m_Jv)^{-1}\left(\int_{I^c} vH_3h_J\right)u_J^{\frac{1}{2}}\|,$$

together with the bound on the norm of D_J to obtain the same result. This doubling assumption may seem more natural to the reader than the assumption we make, until he realizes that it is not even automatic that this doubling assumption is true for $\delta = 0$.

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