Divergence-free multiwavelets on rectangular domains

Joseph Lakey María Cristina Pereyra*

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Abstract

In this paper we construct a family of divergence-free multiwavelets. The construction follows Lemarié's procedure. In the process we find multiresolution analyses (MRA) related by differentiation and integration to a family of biorthogonal MRAs constructed by Hardin and Marasovich. The multiscaling and multiwavelets constructed have symmetries and support properties which allow us to obtain biorthogonal MRAs for the Sobolev space $H_0^1([0,1])$, just by truncating the smoothened scaling functions and wavelets, and keeping those functions that have zero boundary values. These are the building blocks to construct a biorthogonal basis of vector wavelets in $(L^2([0,1]^2))^2$ such that the reconstructing wavelets are divergence-free. These functions constitute a Riesz basis of the L^2-Sobolev space of divergence-free square integrable vector fields on the unit square having tangential boundary components.

1 Introduction

Hardin and Marasovich constructed a family of multiwavelets that have symmetries and short supports. They indicated at the end of their paper [HM] how to obtain a biorthogonal MRA for $L^2([0,1])$ simply by truncating the scaling functions and the wavelets.

We use a modification of the Hardin-Marasovich (HM) MRAs to construct a new biorthogonal pair of MRAs related to that one by differentiation and integration. The main tool in the construction is Strela's Two

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Scale Transform, see [S]; which allows us to reproduce Lemarié's original construction for multiwavelets, see [Le1], [LMP].

The multiscaling and multiwavelets constructed still have ample symmetry and support properties which then allow us to obtain biorthogonal MRAs for the Sobolev space $H_0^1([0,1])$ – that is, the functions compactly supported in $[0,1]$ with square integrable derivatives – simply by truncating the smoothened scaling functions and wavelets, keeping only those functions that have zero boundary values.

P. G. Lemarié-Rieusset introduced a remarkable technique for constructing biorthogonal pairs of compactly supported wavelets in $(L^2(\mathbf{R}^n))^n$, such that the reconstructing wavelets are divergence-free. In addition, he showed that in dimension 2 such wavelets cannot be orthogonal unless one gives up the compact support [Le2]; the authors have extended this result to any dimension [LP]. Lemarié's techniques have been used by K. Urban in [U] to find biorthogonal bases for the space of curl-free vector fields as well. In terms of numerical analysis of incompressible fluids, the only question mark raised by the use of such wavelets is that the divergence-free projection they induce is oblique: it is not the Leray projection. Urban shwed in [U2] that this is not a problem when studying Stoke's problem.

Lemarié's construction is based on the existence of biorthogonal MRAs related by differentiation and integration (see [Le1]), which we will show, exist for the HM biorthogonal multiwavelets. The construction in all dimensions and for form-valued multiwavelets is presented in [LMP] when starting with an orthogonal MRA of multiplicity r. One can reproduce almost verbatim this result in the biorthogonal case, as long as the transition matrices in Strela's two scale transform can be chosen to be M and $-M^*$; as will be the case in the modified Hardin-Marasovich MRA.

Inspired by Lemarié's construction of divergence-free compactly supported wavelets, Urban identified those properties on a wavelet system on $\Omega \subset \mathbf{R}^n$, some open bounded domain with Lipschitz-continuous boundary, that are needed to proceed with the subsequent construction of wavelet bases for the appropiate flux spaces, see [U, p.6].

Restricting ourselves to the case $\Omega = [0,1]^2$ we can proceed as in the $\mathbf{R}^2$ case to obtain biorthogonal systems that satisfy Urban's assumptions. The vector fields we are considering are ideally suited to produce tangential boundary values at the unit square. This very reasonable boundary condition for divergence-free vector fields has not been incorporated into any previous wavelet-Galerkin techniques.

More details about implementation of the HM divergence-free multiwavelets on $[0,1]^2$ will appear in forthcoming work.

2 Multiwavelets

By a multifunction $\Psi(x) = (\psi^1(x), \psi^2(x), \ldots, \psi^r(x))$ we just mean an $\mathbf{R}^r$-valued function on $\mathbf{R}$. Usually operations such as dilation, translation and differentiation will be performed componentwise. For example, we will write $\Psi_{j,k}(x) = 2^{j/2}\Psi(2^j - k) = (\psi^1_{j,k}(x), \ldots, \psi^r_{j,k}(x))$ for all $j, k \in \mathbf{Z}$. Later, we will use the expression $\vec{\Psi}$ to denote a matrix whose entries are multifunctions. Typically, Ψ is regarded as a row vector so that $\Phi^T\Psi$ is a matrix while $\Phi\Psi^T$ is a scalar. Sometimes this notation will be abusive, but it will always be convenient. In this paper we will study a particular family of biwavelets ($r = 2$). Much of what we have to say can be extended to wavelet families of higher multiplicity, but some of the tradeoff between regularity, support and symmetry might be quite specific to the HM family that we consider.

2.1 Biorthogonal multiwavelets

A *biorthogonal biwavelet basis* is a biorthogonal basis generated by a pair of biwavelets $\Psi(x) = (\psi^1(x), \psi^2(x))$, $\tilde{\Psi}(x) = (\tilde{\psi}^1(x), \tilde{\psi}^2(x))$. More precisely, the collection $\{\psi^i_{j,k} = \psi_\lambda\}_{\lambda\in\Lambda}$ is a Riesz basis of $L^2(\mathbf{R})$ with dual basis the collection $\{\tilde{\psi}^i_{j,k} = \tilde{\psi}_\lambda\}_{\lambda\in\Lambda}$, where $\Lambda = \{(i,j,k) : \; i = 1,2; \; j,k \in \mathbf{Z}\}$. We will use $|\lambda| := j$ to denote the *resolution level*. The qualities possessed by these wavelets will include:

- Biorthonormality condition: $\langle\psi_\lambda, \tilde{\psi}_{\lambda'}\rangle = \delta_{\lambda\lambda'}$ for all $\lambda, \lambda' \in \Lambda$, where $\langle.,.\rangle$ denotes the inner product in $L^2(\mathbf{R})$.
- Completeness condition: Given $f \in L^2(\mathbf{R})$ then

$$f = \sum_{\lambda\in\Lambda}\langle f, \tilde{\psi}_\lambda\rangle\psi_\lambda = \sum_{\lambda\in\Lambda}\langle f, \psi_\lambda\rangle\tilde{\psi}_\lambda,$$

  where sums converge in the L^2-sense.
- Riesz basis condition: $\|f\|_2^2 \sim \sum_{\lambda\in\Lambda}|\langle f, \tilde{\psi}_\lambda\rangle|^2 \sim \sum_{\lambda\in\Lambda}|\langle f, \psi_\lambda\rangle|^2$.

2.2 Biorthogonal MRAs with multiple scaling functions

In the "uniwavelet" theory, biorthogonal wavelet bases can be found easily if one has biorthogonal MRAs. The same holds for multiwavelets; the process is more laborious but also more flexible.

A *biorthogonal pair of MRAs* generated by a pair of *biscaling vectors* $\Phi = (\phi^1, \phi^2)$, $\tilde{\Phi} = (\tilde{\phi}^1, \tilde{\phi}^2)$, consists of two MRAs of multiplicity $r = 2$

generated by the corresponding scaling vectors, which satisfy the biorthogonality condition $\langle \phi^i_{j,k}, \tilde{\phi}^l_{j,n} \rangle = \delta_{kn}\delta_{il}$ at each scale j. Moreover the integer translates $\{\phi^i(x-n)\}$, $\{\tilde{\phi}^i(x-n)\}$ form Riesz bases of V_0, and $\tilde{V}_0$ respectively. An orthogonal MRA is the special case $\Phi = \tilde{\Phi}$.

The *basic refinement or dilation equations* read, in vector notation:

$$\Phi(x) = \sum_{n\in\mathbf{Z}} H_n \Phi(2x-n), \qquad \tilde{\Phi}(x) = \sum_{n\in\mathbf{Z}} \tilde{H}_n \tilde{\Phi}(2x-n),$$

where H_n and $\tilde{H}_n$ are 2×2 matrices for each $n \in \mathbf{Z}$. If all but a finite number are zero the multiscaling functions will have compact support. We shall assume this to be the case in the sequel.

By the scaling and Riesz basis properties we have two increasing sequences of closed subspaces $\{V_j\}, \{\tilde{V}_j\}$ for $j \in \mathbf{Z}$, which are nested: $V_j \subset V_{j+1}$ and $\tilde{V}_j \subset \tilde{V}_{j+1}$ and the scaling property $f(x) \in V_j$ (or $\tilde{V}_j$) if and only if $f(2x) \in V_{j+1}$ (or $\tilde{V}_{j+1}$). In fact, due to biorthogonality we can write as direct sums (not necessarily orthogonal) the spaces $V_{j+1} = V_j \oplus W_j$, $\tilde{V}_{j+1} = \tilde{V}_j \oplus \tilde{W}_j$, where $V_j \perp \tilde{W}_j$ and $\tilde{V}_j \perp W_j$. Finally, we require that the union of either chain is dense in $L^2(\mathbf{R})$ and that the intersection of either chain is the trivial subspace.

On Fourier side the dilation equations become:

$$\hat{\Phi}(\xi) = H(\xi/2)\hat{\Phi}(\xi/2), \qquad \hat{\tilde{\Phi}}(\xi) = \tilde{H}(\xi/2)\hat{\tilde{\Phi}}(\xi/2), \tag{1}$$

where $H(\xi)$ and $\tilde{H}(\xi)$ are the *refinement masks* corresponding to the multiscaling functions Φ, $\tilde{\Phi}$ respectively:

$$H(\xi) = \frac{1}{2}\sum_{n\in\mathbf{Z}} H_n e^{-in\xi}, \qquad \tilde{H}(\xi) = \frac{1}{2}\sum_{n\in\mathbf{Z}} \tilde{H}_n e^{-in\xi}.$$

We have defined the Fourier transform for Schwartz functions f by

$$\hat{f}(\xi) = \int_{\mathbf{R}} f(x) e^{-ix\xi} dx,$$

and on $L^2(\mathbf{R})$ by continuity.

The refinement masks are matrix trigonometric polynomials. Iterating the first equation in (1) we get:

$$\hat{\Phi}(\xi) = H\big(\frac{\xi}{2}\big) H\big(\frac{\xi}{4}\big)\hat{\Phi}\big(\frac{\xi}{4}\big) = \ldots = \prod_{j=1}^{\infty} H\big(\frac{\xi}{2^j}\big)\hat{\Phi}(0). \tag{2}$$

The same holds for $\hat{\tilde{\Phi}}$ and $\tilde{H}$. By using tricks that are familiar in the uniwavelet setting, the biorthogonality condition can be recast in terms of the refinement masks. To do so, we shall first make the substitution $z = e^{-i\xi}$ so that the entries of $H, \tilde{H}$ become actual polynomials of the complex variable z. A word of warning: we will revert to thinking of $H, \tilde{H}$ as functions of ξ when formulas involve both filters and scaling or wavelet vectors, but in formulas involving only the filters we will think of them as functions of z. In any case, the two points of importance are $\xi = 0$ $(z = 1)$ and $\xi = \pi$ $(z = -1)$ so our normalization of the Fourier transform should prevent any confusion as to whether we are thinking of the masks of functions of ξ or of z. Having said this, the condition of biorthogonality in terms of the refinement masks becomes:

$$H(z)\tilde{H}^*(z) + H(-z)\tilde{H}^*(-z) = I.$$

Remark: From now on, we assume that $H(0)$ has a simple eigenvalue $\lambda = 1$, and all its other eigenvalues are strictly less that 1 in absolute value. This condition ensures uniform convergence of the infinite product (2), see [CDP] and [HC]. Notice that $\hat{\Phi}(0)$ must be an eigenvector of $H(0)$ corresponding to the simple eigenvalue $\lambda = 1$.

As mentioned, in the uniwavelet case it is well known how to find biorthogonal wavelet bases belonging to biorthogonal MRAs. The case of multiwavelets is a little bit more involved. The theory grew out of specific cases. The first multiwavelet construction is due to Alpert and Hervé. The functions were piecewise linear, but discontinuous. Using fractal interpolation Geronimo, Hardin, and Massopust [GHM] succeeded in constructing orthogonal continuous biscaling functions with short support, symmetry and second approximation order. A little later the HM wavelets were developed as a natural conceptual framework for the GHM wavelets. At about the same time, Strela [S] developed a more systematic theory for constructing multiwavelets from a given multi MRA, the approach being to complete a perfect reconstruction filter bank given the scaling filters $(H, \tilde{H})$. We will apply some of his ideas in the context of the HM wavelets. The problem reduces to finding a pair of matrix valued trigonometric polynomials (the multi-filters) $(F, \tilde{F})$ such that the following matrix equations hold:

$$\begin{aligned} H(z)\tilde{H}^*(z) + H(-z)\tilde{H}^*(-z) &= I \\ F(z)\tilde{F}^*(z) + F(-z)\tilde{F}^*(-z) &= I \\ H(z)\tilde{F}^*(z) + H(-z)\tilde{F}^*(-z) &= 0 \\ F(z)\tilde{H}^*(z) + F(-z)\tilde{H}^*(-z) &= 0\,. \end{aligned} \qquad \begin{array}{l}\text{Biorthogonality} \\ \text{Conditions (BC)}\end{array}$$

On Fourier side, the multiwavelets will satisfy the following equations:

$$\hat{\Psi}(\xi) = F(\xi/2)\hat{\Phi}(\xi/2), \qquad \hat{\tilde{\Psi}}(\xi) = \tilde{F}(\xi/2)\hat{\tilde{\Phi}}(\xi/2).$$

In the uniwavelet case the standard intertwined *alternating flip* choice does the job. Though the trick breaks down for multiwavelets because matrices do not commute, with some more work suitable high pass multifilters can be found. But this is not the primary focus of the current work; rather, we will take the Hardin and Marosovich [HM] family as our starting point. By suitably modifying certain parameters we will build dual families that are amenable to the multiwavelet version of Lemarié's smoothening and roughening technique.

2.3 Modified Hardin-Marasovich (HM) filters and scaling functions

Hardin and Marasovich provided a somewhat more unified view of the GHM construction of scaling functions via fractal interpolation. By construction, their spaces V_0 and $\tilde{V}_0$ contain piecewise linear continuous functions with nodes at the integers. We will adopt the terminology of phrasing this local reproduction of linear functions as saying that the MRA *has approximation order 2*. In what follows we will essentially present the HM setup, but with a modification of the HM parameter "δ" (see Appendix B). This modification slightly simplifies the description of the dual scaling filters, but in a crucial way that permits us to smoothen and roughen the HM pairs consistent with their parametrization of symmetric biorthogonal fractal interpolation MRA's. They considered nonsymmetric filters also, but those are not relevant to our current goals. The modified symmetric HM filters depend on a parameter s which will be explicit in Appendix A, where we will remind the reader the clever geometric argument that led to the construction of these examples. Continuity of the scaling vectors requires $|s| < 1$, and for Lipschitz-α continuity one needs $|s| < 2^{-\alpha}$, as is shown in Appendix A.

Modified HM multiscaling filter (see [HM Sec 3.3, p.21]).

The filter corresponding to the modified HM biscaling function with parameter value s is:

$$H_s(z) = C_{-2}(s)z^{-2} + C_{-1}(s)z^{-1} + C_0(s) + C_1(s)z,$$

where the coefficients are given by:

$$C_{-2}(s) \quad = \quad \frac{1}{24}\begin{bmatrix} 0 & -(1+2s)\sqrt{2} \\ 0 & 0 \end{bmatrix},$$

$$
\begin{aligned}
C_{-1}(s) &= \frac{1}{24}\begin{bmatrix} 8s-2 & (5-2s)\sqrt{2} \\ 0 & 0 \end{bmatrix}, \\
C_0(s) &= \frac{1}{24}\begin{bmatrix} 12 & (5-2s)\sqrt{2} \\ 0 & 8+4s \end{bmatrix}, \\
C_1(s) &= \frac{1}{24}\begin{bmatrix} 8s-2 & -(1+2s)\sqrt{2} \\ (8-8s)\sqrt{2} & 8+4s \end{bmatrix}. \qquad (3)
\end{aligned}
$$

Modified HM dual multiscaling filter

With the parametrization above, and the dual parameter $\tilde{s} = \frac{1+2s}{5s-2}$ defined as in [HM], the dual filter satisfies $\tilde{H}_s = H_{\tilde{s}}$ Again, we refer to Appendix B for more details.

The dual filter $\tilde{H}_s$ is given explicitly in terms of s by:

$$\tilde{H}_s(z) = \tilde{C}_{-2}(s)z^{-2} + \tilde{C}_{-1}(s)z^{-1} + \tilde{C}_0(s) + \tilde{C}_1(s)z,$$

where the coefficients are given by $\tilde{C}_k(s) = C_k(\tilde{s})$:

$$
\begin{aligned}
\tilde{C}_{-2}(s) &= \frac{1}{24(5s-2)}\begin{bmatrix} 0 & -9s\sqrt{2} \\ 0 & 0 \end{bmatrix}, \\
\tilde{C}_{-1} &= \frac{1}{24(5s-2)}\begin{bmatrix} 6s+12 & (21s-12)\sqrt{2} \\ 0 & 0 \end{bmatrix}, \\
\tilde{C}_0(s) &= \frac{1}{24(5s-2)}\begin{bmatrix} 60s-24 & (21s-12)\sqrt{2} \\ 0 & 48s-12 \end{bmatrix}, \\
\tilde{C}_1(s) &= \frac{1}{24(5s-2)}\begin{bmatrix} 6s+12 & -9s\sqrt{2} \\ (24s-24)\sqrt{2} & 48s-12 \end{bmatrix}. \qquad (4)
\end{aligned}
$$

About the parameter s: If we want both multiscaling functions to be at least continuous then we need both $|s|, |\tilde{s}| < 1$ which implies $-1 < s < \frac{1}{7}$. If, in addition, we want one multiscaling function to be Lipchitz continuous, we need $|s| \leq 1/2$. Note that, then, $|\tilde{s}| \leq 1/2$ also. See Appendix A. When $s = -1/5$ then $s = \tilde{s}$ hence $H_s = H_{\tilde{s}}$, and we encounter the orthogonal MRA constructed in [GHM].

Biorthogonality: It can be checked that $H_s, H_{\tilde{s}}$ satisfy the biorthogonality condition: $H_s(z)H_{\tilde{s}}^*(z) + H_s(-z)H_{\tilde{s}}^*(-z) = I$ (see Appendix C). This is best accomplished with the aid of a symbolic algebra tool.

Modified HM multiwavelet filter (see [HM Sec 3.3 p.22]).

In the context of the aforementioned modification, the *HM completion* of the multiwavelet filter bank produces the wavelet filter

$$F_s(z) = D_{-2}(s)z^{-2} + D_{-1}(s)z^{-1} + D_0(s) + D_1(s)z,$$

where the coefficients are given by:

$$\begin{aligned}
D_{-2}(s) &= \frac{1}{24}\begin{bmatrix} 0 & -(1+2s)\sqrt{2} \\ 0 & -2-4s \end{bmatrix}, \\
D_{-1}(s) &= \frac{1}{24}\begin{bmatrix} 8s-2 & (5-2s)\sqrt{2} \\ (8s-2)\sqrt{2} & 10-4s \end{bmatrix}, \\
D_{0}(s) &= \frac{1}{24}\begin{bmatrix} -12 & (5-2s)\sqrt{2} \\ 0 & 4s-10 \end{bmatrix}, \\
D_{1}(s) &= \frac{1}{24}\begin{bmatrix} 8s-2 & -(1+2s)\sqrt{2} \\ (2-8s)\sqrt{2} & 2+4s \end{bmatrix}.
\end{aligned} \tag{5}$$

Modified HM dual multiwavelet filter

As for the scaling multifilters, the dual wavelet multifilter is found by replacing s by $\tilde{s} = \frac{1+2s}{5s-2}$ above, that is, $\tilde{F}_s = F_{\tilde{s}}$. Again this differs with the original HM dual multiwavelet filter.

The dual wavelet filter $\tilde{F}_s$ is, in terms of s:

$$\tilde{F}_s(z) = \tilde{D}_{-2}(s)z^{-2} + \tilde{D}_{-1}(s)z^{-1} + \tilde{D}_0(s) + \tilde{D}_1(s)z,$$

where the coefficients are given by $\tilde{D}_k(s) = D_k(\tilde{s})$:

$$\begin{aligned}
\tilde{D}_{-2}(s) &= \frac{1}{24(5s-2)}\begin{bmatrix} 0 & -9s\sqrt{2} \\ 0 & -18s \end{bmatrix}, \\
\tilde{D}_{-1}(s) &= \frac{1}{24(5s-2)}\begin{bmatrix} 6s+12 & (21s-12)\sqrt{2} \\ (6s+12)\sqrt{2} & 42s-24 \end{bmatrix}, \\
\tilde{D}_{0}(s) &= \frac{1}{24(5s-2)}\begin{bmatrix} 24-60s & (21s-12)\sqrt{2} \\ 0 & 24-42s \end{bmatrix}, \\
\tilde{D}_{1}(s) &= \frac{1}{24(5s-2)}\begin{bmatrix} 6s+12 & -9s\sqrt{2} \\ -(6s+12)\sqrt{2} & 18s \end{bmatrix}.
\end{aligned} \tag{6}$$

With this filterbank completion, all of the conditions of biorthogonality are satisfied (see Appendix C) and the following result holds.

Theorem 1 (Hardin-Marasovich) *Suppose $-1 < s < \frac{1}{7}$, let $\tilde{s} = \frac{1+2s}{5s-2}$. Let $\Phi_s = (\phi_s^1, \phi_s^2)$, $\tilde{\Phi}_s = \Phi_{\tilde{s}} = (\phi_{\tilde{s}}^1, \phi_{\tilde{s}}^2)$ be the modified HM multiscaling functions corresponding to the filters H_s and $H_{\tilde{s}}$. Let $\{V_j\}$ and $\{\tilde{V}_j\}$ be the corresponding MRA's generated by them. Then:*

(a) *$\Phi_s, \Phi_{\tilde{s}}$ are continuous biorthogonal multiscaling functions.*

(b) $\phi_s^1, \phi_{\tilde{s}}^1$ *are supported on* $[-1,1]$ *and are symmetric about* $x = 0$ *while* $\phi_s^2, \phi_{\tilde{s}}^2$ *are supported on* $[0,1]$ *and are symmetric about* $x = 1/2$.

(c) *The hat function* $h(x) = (1-|x|)\chi_{[-1,1]}(x)$ *is in both* V_0 *and* $\tilde{V}_0$. *Thus the scaling vectors have* approximation order 2.

(d) *The associated biorthogonal multiwavelets* $\Psi_s, \tilde{\Psi}_s = \Psi_{\tilde{s}}$, *are continuous and supported in* $[-1,1]$ *with filters* F_s *and* $F_{\tilde{s}}$.

(e) $\psi_s^1, \psi_{\tilde{s}}^1$ *are symmetric about* $x = 0$ *while* $\psi_s^2, \psi_{\tilde{s}}^2$ *are antisymmetric about* $x = 0$.

Because the wavelets have compact support they can be written as finite linear combinations of the components of the multiscaling functions and their integer translates. Therefore the multiwavelets will have same regularity as the multiscaling functions. In particular, they will be Lipschitz continuous whenever $|s| < 1/2$ (Appendix A).

The special case $s = 0$, $\tilde{s} = -1/2$, is illustrated in Figure 1. Note that the functions plotted are shifts by two units of the actual HM scaling and wavelet functions.

Eigenvalue 1: In the theory of uniwavelets the filter corresponding to the scaling function is *low pass*, in particular $H(1) = 1$ ($z = e^{-i\xi} = 1$ corresponds to $\xi = 0$). The wavelet filter is *high pass*: $F(1) = 0$. These conditions reflect the facts that $\hat{\phi}(0) = 1$ and $\hat{\psi}(0) = 0$. In the multiwavelet case the analogues of these conditions read: $H_s(1)\Phi^T(0) = \hat{\Phi}^T(0)$ and $F(1)\hat{\Phi}^T(0) = \hat{\Psi}^T(0)$, see Remark in page 5. In plain English: $\hat{\Phi}^T(0)$ *is an eigenvector with simple eigenvalue* 1 *of* $H(1)$. All other eigenvalues must be strictly less than 1 in absolute value to ensure uniqueness and convergence of Φ. To ensure that Φ provides aproximation order 2, further conditions such as $\hat{\Phi}(0)H_s(-1) = 0$ are required [S, Thm 3.1.1, p. 38]. Such conditions naturally generalize those in the scalar case where m-th approximation order implies m zeros of $H(z) = H(\xi)$ at $z = -1$ ($\xi = \pi$).

In the symmetric HM case it turns out that $H_s(1)$ is symmetric:

$$H_s(1) = \frac{1}{3}\begin{bmatrix} 1+2s & (1-s)\sqrt{2} \\ (1-s)\sqrt{2} & 2+s \end{bmatrix};$$

with characteristic polynomial $(\lambda - 1)(\lambda - s)$. Hence $H_s(1)$ has eigenvalues $\lambda = 1$ and $\lambda = s$. It is not difficult to check that the vector $u = (1, \sqrt{2})^T$ is a right and left eigenvector for the eigenvalue $\lambda = 1$ *independently of* s. Notice also that $u^T H_s(-1) = 0$ and that

$$F_s(1) = \frac{1}{3}\begin{bmatrix} 2s-2 & (1-s)\sqrt{2} \\ 0 & 0 \end{bmatrix};$$

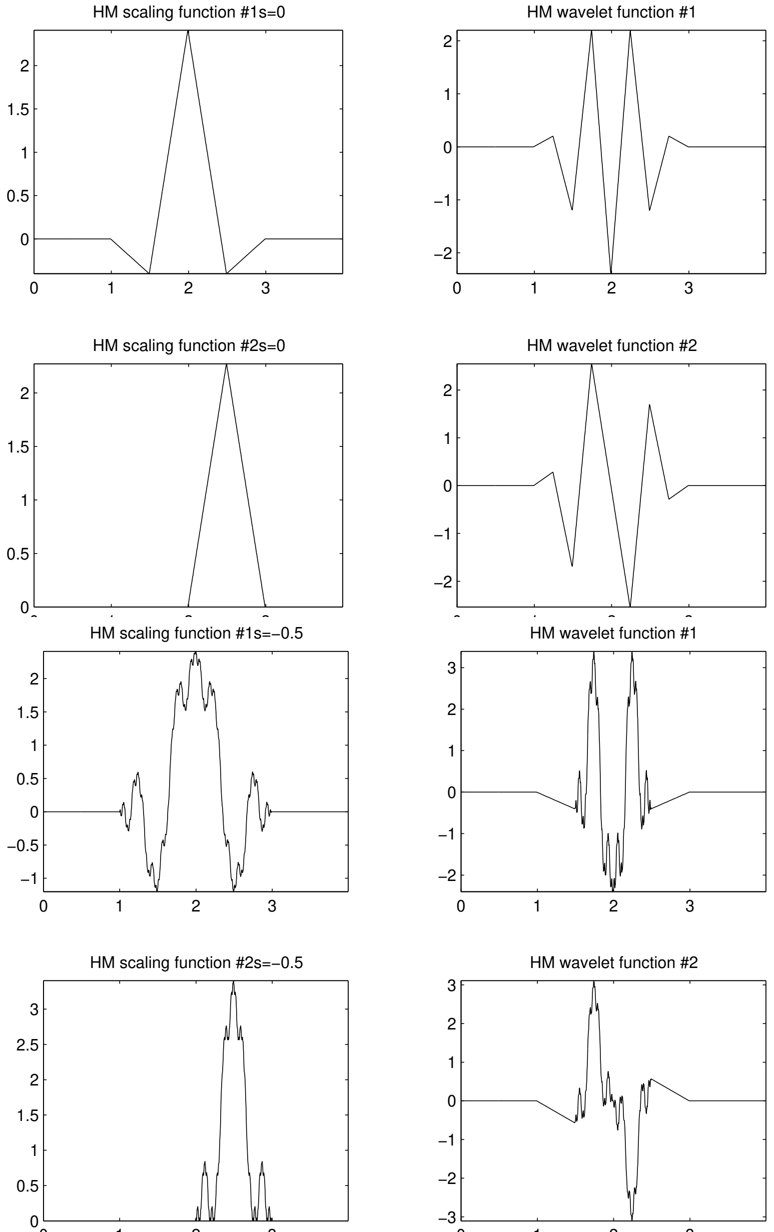

Figure 1. Plot of HM scaling and wavelet functions, $s = 0$, $\tilde{s} = -1/2$.

hence $F_s(1)u = 0$ for all s, therefore $\hat{\Psi}(0) = 0$, and all the components of the multiwavelet have mean value zero as in the scalar case. In particular the symmetric components will have mean value zero on the interval $[0, 1]$. This will be important for us later.

3 MRA's related by differentiation and integration

3.1 Strela's Two Scale Transform and differentiation

Strela introduced the *two scale transform* as a tool to influence approximation order and regularity of a given multiscaling funtion Φ with refinement mask or filter H. His analysis is carried out at the level of the filters. The two scale transform enables the construction of new filters from old ones. We are interested in a particular transform that permits us to construct new biorthogonal multiscaling filters corresponding to a pair of biorthogonal multiscaling functions related to a given biorthogonal pair by differentiation and integration. This property is a crucial ingredient for building divergence-free multiwavelets, as Lemarié first observed in [Le1].

From our perspective, Strela's two scale transform can be viewed as an extension of Lemarié's *smoothening* or *roughening* technique to the multiscaling context. As the regularity is increased or decreased, an order of approximation is also gained or lost. In the HM case, Strela's procedure goes as follows:

Smoothening ([S, Lemma 3.4.2 p.49]: The HM filter $H_s(z)$, $|s| < 1$ has approximation order 2, and the scaling vector Φ_s is continuous. The matrix $H_s(1)$ has a simple eigenvalue $\lambda = 1$ with eigenvector $u = (1, \sqrt{2})^T$. The main step is to find a transition matrix $A(z)$ such that:

- $A(z)$ is invertible for all $z \neq 1$, (s1)
- $A(1)u = 0$, and $\lambda = 0$ is a simple eigenvalue. (s2)

Once A is found, the new filter given by the two scale transform:

$$H_s^+(z) = \frac{1}{2} A(z^2) H_s(z) A^{-1}(z)$$

has one more order of approximation (that is 3), and the associated multiscaling function satisfies the frequency equation: $i\xi\hat{\Phi}_s^+(\xi) = A(\xi)\hat{\Phi}_s(\xi)$. Upon taking inverse Fourier transforms, one then has

$$D\Phi_s^+ = -T_A \Phi_s,$$

where T_A is a convolution operator with matrix valued convolution kernel the inverse Fourier transform of $A(\xi)$. The matrix A will be chosen to be a matrix trigonometric polynomial, therefore T_A will be a generalized translation given by multiplication by the matrix polynomial on the shift $Sf(x) = f(x-1)$ found by replacing z by S in the transition matrix A. The new multiscaling function Φ_s^+ will thus be compactly supported and continuously differentiable.

Roughening [S, Thm 3.4.3 p. 50]: The HM filter $H_s(z)$, $|s| < \frac{1}{2}$ has approximation order 2, and the scaling vector Φ_s is Lipschitz continuous. The matrix $H_s(1)$ has a simple eigenvalue $\lambda = 1$ with left eigenvector $u = (1, \sqrt{2})^T$ (remember $H_s(1)$ is symmetric). The goal is to find a transition matrix $M(z)$ such that:

- $M(z)$ is invertible for all $z \neq 1$, (r1)
- $u^T M(1) = 0$, and $\lambda = 0$ is a simple eigenvalue. (r2)

Once M is found, the new filter given by the inverse two scale transform:

$$H_s^-(z) = 2M^{-1}(z^2)H_s(z)M(z)$$

has one less order of approximation (that is 1), and the associated multiscaling function satisfies the frequency equation: $i\xi\hat{\Phi}_s(\xi) = M(\xi)\hat{\Phi}_s^-(\xi)$. Upon inverse Fourier transforming, one has

$$D\Phi_s = -T_M\Phi_s^-,$$

where T_M is a convolution operator with matrix valued convolution kernel the inverse Fourier transform of $M(\xi)$. M will be chosen to be a matrix trigonometric polynomial, therefore T_M will be a generalized translation, and the new multiscaling funtion Φ_s^- will be piecewise continuous.

Because the matrix $H_s(1)$ is symmetric, once the roughening matrix $M(z)$ is found, we can choose $A(z) = -M^*(z)$ as the smoothening matrix. Notice that with this choice $T_A = -T_M^*$ (here T^* means the Hilbert space adjoint of T). A pair M, $-M^*$ that works for a particular H_s, will work for the whole scale s. In fact this same pair will work for any biorthogonal completion of H_s. This can be deduced from the symmetry of H_s, the observation that $uH_s(-1) = 0$ and the biorthogonality conditions.

In principle, when computing the two scale transform with given transition matrix polynomial M, the output does not have to be a matrix polynomial since the inverse M^{-1} enters into the transform. To insure *compact* *support* for either the roughened or smoothed multiscaling function the entries of $M(z)$ will be required to be linear in z, see [S, Lemma 3.6.2 p.54]. In particular $M(z) = M_0 + M_1 z$ (here M_i are 2×2 scalar matrices). Then $M^*(z) = M_0^* + M_1^* z^{-1}$ and the corresponding convolution operators are given by

$$T_M = M_0 + M_1 S, \quad T_M^* = M_0^* + M_1^* S^{-1} = T_{M^*}.$$

Here we are using the fact that $S^* = S^{-1}$, and

$$M_1^* S^{-1} = S^{-1} M_1^* = \begin{pmatrix} aS^{-1} & bS^{-1} \\ cS^{-1} & dS^{-1} \end{pmatrix}, \qquad M_1^* = \begin{pmatrix} a & b \\ c & d \end{pmatrix};$$

see Appendix D.

Commutation relations. The use of adjoints plays an important role in a summation by parts argument that is used to show that differentiation commutes with oblique MRA projections; for example: $D\tilde{P}_0^+ = \tilde{P}_0 D$, where we denote by $\tilde{P}_0 f = \sum_k \langle f, \Phi_{0,k}\rangle \tilde{\Phi}_{0,k}$ (the oblique projection onto $\tilde{V}_0$), and $\tilde{P}_0^+ f = \sum_k \langle f, \Phi_{0,k}^-\rangle \tilde{\Phi}_{0,k}^+$ (the oblique projection onto $\tilde{V}_0^+$). Here is the argument:

$$\tilde{P}_0 Df = \sum_k \langle Df, \Phi_{0,k}\rangle \tilde{\Phi}_{0,k} = \sum_k \langle f, -D\Phi_{0,k}\rangle \tilde{\Phi}_{0,k} = \sum_k \langle f, T_M \Phi_{0,k}^-\rangle \tilde{\Phi}_{0,k} =$$

$$= \sum_k \langle f, \Phi_{0,k}^-\rangle T_{M^*} \tilde{\Phi}_{0,k} = \sum_k \langle f, \Phi_{0,k}^-\rangle D\tilde{\Phi}_{0,k}^+ = D\tilde{P}_0^+ f.$$

The commutation relations are crucial in the construction of divergence-free wavelets coming from an MRA. They will guarantee that the MRA structure persists in the space of divergence-free vector fields.

We also want to preserve *symmetries*. Strela laid down a criterion to that end as well, see [S, Lemma 3.5.2 p. 52]:

Symmetries: Suppose $D\Phi = -T_M \Phi^-$, where both Φ and Φ^- are refinable and have components which are symmetric or antisymmetric about the points t_1, t_2 and t_1^-, t_2^- respectively. Then necessarily:

$$M(z) = E(z) M(z^{-1}) E_-^{-1}(z),$$

where

$$E(z) = \begin{pmatrix} \pm z^{2t_1} & 0 \\ 0 & \pm z^{2t_2} \end{pmatrix}, \qquad E_-(z) = \begin{pmatrix} \pm z^{2t_1^-} & 0 \\ 0 & \pm z^{2t_2^-} \end{pmatrix}.$$

The plus or minus signs are dictated by the symmetry or antisymmetry of the corresponding components of $D\Phi$ and Φ^-.

3.2 Smoothened and Roughened HM multiscaling filters

Consider the matrix

$$M(z) = \begin{pmatrix} 0 & 2\sqrt{2} \\ 1-z & -1-z \end{pmatrix}. \tag{7}$$

It is not hard to check that M satisfies (r1) and (r2) on p.12; moreover, $M(z) = E(z)M(z^{-1})E_-^{-1}(z)$, where

$$E(z) = \begin{pmatrix} -1 & 0 \\ 0 & -z \end{pmatrix}, \qquad E_-(z) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Theorem 2 *Let H_s, $H_{\tilde{s}}$ be HM scaling filters with $\tilde{s} = \frac{1+2s}{5s-2}$, $-\frac{1}{2} < s < \frac{1}{7}$, so that $-1 < \tilde{s} < 0$, hence Φ_s is Lipschitz continuous, and $\Phi_{\tilde{s}}$ is continuous. Let $M(z)$ be given by (7) and, according to Strela's criteria, set:*

$$H_s^-(z) = 2M^{-1}(z^2)H_s(z)M(z), \qquad H_{\tilde{s}}^+(z) = \frac{1}{2}M^*(z^2)H_{\tilde{s}}(z)(M^*)^{-1}(z).$$

Then the new filters satisfy the biorthogonality condition:

$$H_{\tilde{s}}^+(z)(H_s^-)^*(z) + H_{\tilde{s}}^+(-z)(H_s^-)^*(-z) = I.$$

The corresponding smoothened and roughened multiscaling functions $\Phi_{\tilde{s}}^+$ and Φ_s^- generate a biorthogonal MRA in $L^2(\mathbf{R})$. Moreover they satisfy:

(a) $D\Phi_s = -T_M\Phi_s^-$, $D\Phi_{\tilde{s}}^+ = T_M^*\Phi_s$.

(b) *Both components of Φ_s^- are piecewise continuous and supported on $[-1,1]$; the first component $\phi_s^{1,-}$ is symmetric about 0, the second $\phi_s^{2,-}$ is antisymmetric about 0. Moreover, let $\alpha = \phi_s^{1,-}\chi_{[-1,0]}$ and $\beta = \phi_s^{1,-}\chi_{[0,1]}$, then $\phi_s^{1,-} = \alpha + \beta$ and $\phi_s^{2,-} = \alpha - \beta$.*

(c) *Both components of $\Phi_{\tilde{s}}^+$ are supported on $[-1,1]$. The first component $\phi_{\tilde{s}}^{1,+}$ is symmetric about 0. The second component $\phi_{\tilde{s}}^{2,+}$ is antisymmetric about 0.*

Proof: The biorthogonality condition of the new filters is an immediate consequence of the biorthogonality condition of the filters H_s, $H_{\tilde{s}}$. The differentiability property (a) is a consequence of Strela's smoothening and roughening trick. The proofs of the symmetry properties, parts (b) and (c) of the theorem, are postponed to Appendix D. □

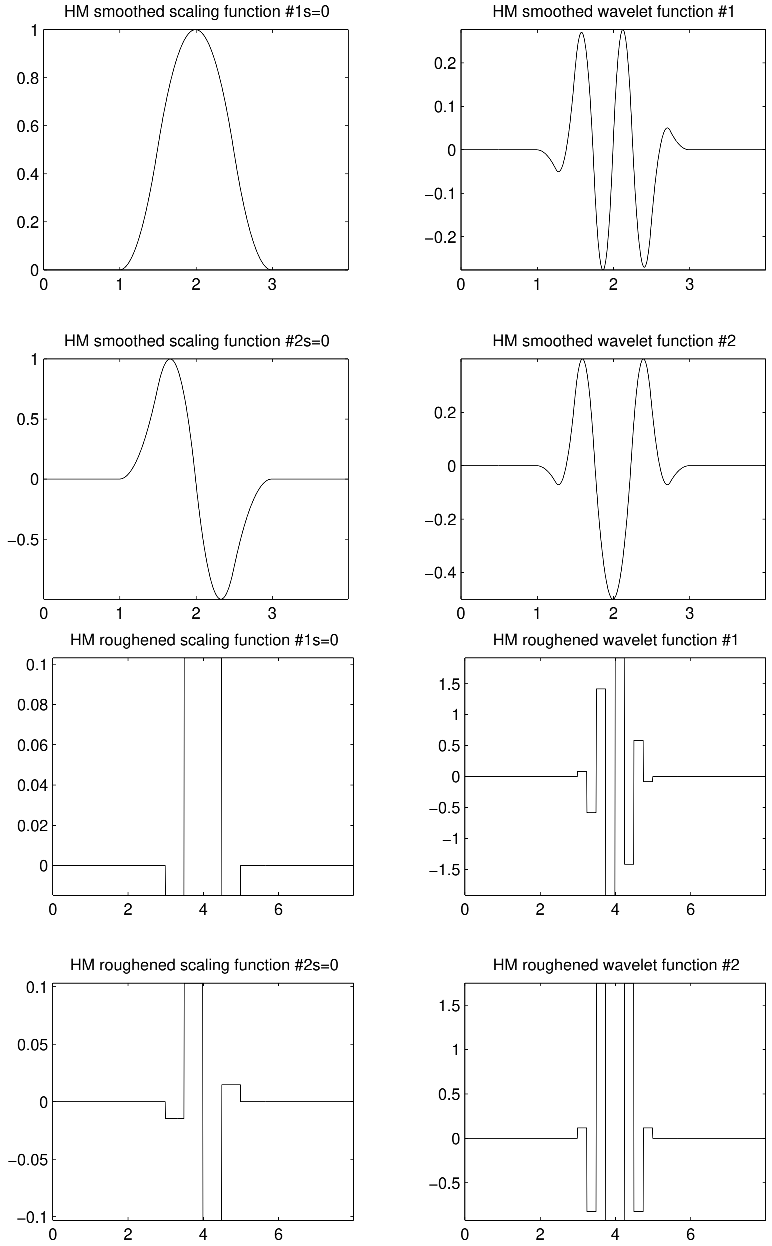

Figure 2. Plots of smoothened and roughened HM's, $s = 0$.

The case where $s = 0$ is illustrated in Figure 2. Again the plots are shifted versions of the actual functions: the smoothened by two units, the roughened by four.

Transition Matrices:

For the record, the transition matrices M, M^* and their inverses are:

$$M(z) = \begin{pmatrix} 0 & 2\sqrt{2} \\ 1-z & -1-z \end{pmatrix},$$

$$M^{-1}(z) = \frac{-1}{2\sqrt{2}(1-z)} \begin{pmatrix} -1-z & -2\sqrt{2} \\ z-1 & 0 \end{pmatrix};$$

$$M^*(z) = \begin{pmatrix} 0 & 1-\frac{1}{z} \\ 2\sqrt{2} & -1-\frac{1}{z} \end{pmatrix},$$

$$(M^*)^{-1}(z) = \frac{-1}{2\sqrt{2}(1-\frac{1}{z})} \begin{pmatrix} -1-\frac{1}{z} & \frac{1}{z}-1 \\ -2\sqrt{2} & 0 \end{pmatrix};$$

Again, with the aid of symbolic calculator we can compute the new filters in terms of s.

Smoothened HM multiscaling filter:

$$H_s^+(z) = C_{-1}^+(s)z^{-1} + C_0^+(s) + C_1^+(s)z,$$

where the coefficients are given by:

$$C_{-1}^+(s) = \begin{bmatrix} \frac{1}{4} & \frac{1-s}{4(5s-2)} \\ \frac{3}{8} & \frac{4-s}{4(5s-2)} \end{bmatrix}, \tag{8}$$

$$C_0^+(s) = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{4} \end{bmatrix}, \quad C_1^+(s) = \begin{bmatrix} \frac{1}{4} & \frac{s-1}{4(5s-2)} \\ -\frac{3}{8} & \frac{4-s}{4(5s-2)} \end{bmatrix}.$$

Note, in particular, that the entries are still symmetrical or antisymmetrical, as predicted.

Roughened HM multiscaling filter:

$$H_s^-(z) = C_{-2}^-(s)z^{-2} + C_{-1}^-(s)z^{-1} + C_0^-(s) + C_1^-(s)z + C_2^-(s)z^2,$$

where the coefficients are given by:

$$C_{-2}^{-}(s) = \frac{1}{24}\begin{bmatrix} -1-2s & (1+2s) \\ -1-2s & 1+2s \end{bmatrix}, \quad C_{-1}^{-}(s) = \frac{1}{24}\begin{bmatrix} 6 & (20s-8) \\ 6 & 20s-8 \end{bmatrix},$$

$$C_{0}^{-}(s) = \frac{1}{24}\begin{bmatrix} 14+4s & 0 \\ 0 & 14+4s \end{bmatrix}, \quad C_{1}^{-}(s) = \frac{1}{24}\begin{bmatrix} 6 & (8-20s) \\ -6 & 20s-8 \end{bmatrix},$$

$$C_{2}^{-}(s) = \frac{1}{24}\begin{bmatrix} -1-2s & (-1-2s) \\ 1+2s & 1+2s \end{bmatrix}. \tag{9}$$

Once more the entries are all either symmetrical or antisymmetrical.

3.3 Smoothened and Roughened HM multiwavelets filters

In [LMP] a simple construction was presented for the roughened and smoothened multiwavelets, so that the biorthogonality conditions remain valid and the old wavelets are related to the new by differentiation and integration. The starting point there was an orthonormal MRA, but the construction still works in the HM biorthogonal case.

The biorthogonal multiwavelets Ψ_s and $\Psi_{\tilde{s}}$ have the same smoothness as the multiscaling functions, so we will smoothen the roughest (that is $\Psi_{\tilde{s}}$), and roughen the smoothest (that is Ψ_s), as we did with the scaling functions. The simplest choise of high-pass filters, as was shown in [LMP] for the orthogonal case, still works in this case.

Theorem 3 *Let F_s, $F_{\tilde{s}}$ be HM wavelet filters with $\tilde{s} = \frac{1+2s}{5s-2}$, $-\frac{1}{2} < s < \frac{1}{7}$, so that $-1 < \tilde{s} < 0$, hence Ψ_s is Lipschitz continuous, and $\Psi_{\tilde{s}}$ is continuous. Let*

$$F_s^{-}(z) = 2F_s(z)M(z), \qquad F_{\tilde{s}}^{+}(z) = \frac{1}{2}F_{\tilde{s}}(z)(M^*)^{-1}(z); \tag{10}$$

where $M(z)$ is the transition matrix (7) used in the construction of $(H_s^-,H_{\tilde{s}}^+)$. Then the new filters $(H_s^-, H_{\tilde{s}}^+)$, $(F_s^-, F_{\tilde{s}}^+)$, satisfy the biorthogonality conditions:

$$\begin{aligned} H_s^-(z)(H_{\tilde{s}}^+)^*(z) + H_s^-(-z)(H_{\tilde{s}}^+)^*(-z) &= I \\ F_s^-(z)(F_{\tilde{s}}^+)^*(z) + F_s^-(-z)(F_{\tilde{s}}^+)^*(-z) &= I \\ H_s^-(z)(F_{\tilde{s}}^+)^*(z) + H_s^-(-z)(F_{\tilde{s}}^+)^*(-z) &= 0 \\ F_s^-(z)(H_{\tilde{s}}^+)^*(z) + F_s^-(-z)(H_{\tilde{s}}^+)^*(-z) &= 0\,. \end{aligned}$$

The corresponding multiwavelets form a Riesz basis and they are related by differentiation in the simplest possible way, namely:

$$D\Psi_{\tilde{s}}^{+} = \Psi_{\tilde{s}}, \qquad D\Psi_{s} = -\Psi_{s}^{-}.$$

The supports of the new wavelets remain unchanged, that is they are supported on the interval $[-1,1]$. Therefore the new filters must be matrix trigonometric polynomials. The symmetries persist, although whatever component that was symmetric becomes antisymmetric and viceversa.

Proof: Since the filters $(H_s, H_{\tilde{s}})$, $(F_s, F_{\tilde{s}})$ satisfy the biorthogonality condictions (BC), then the new filters satisfy their corresponding biorthogonality conditions.

Using the differentiability properties of the scaling function we get:

$$\hat{\Psi}_{\tilde{s}}^{+}(2\xi) = F_{\tilde{s}}^{+}(\xi)\hat{\Phi}_{\tilde{s}}^{+}(\xi) = \frac{1}{2}F_{\tilde{s}}(\xi)(M^{*})^{-1}(\xi)\frac{-M^{*}(\xi)}{i\xi}\hat{\Phi}_{\tilde{s}}(\xi) = \frac{-1}{i2\xi}\hat{\Psi}_{\tilde{s}}(2\xi),$$

which implies $D\Psi_{\tilde{s}}^{+} = \Psi_{\tilde{s}}$. Similarly, $D\Psi_{s} = -\Psi_{s}^{-}$. The Riesz basis property is a consequence of the classical Vaguelette Lemma, see [M, p270]. $\square$

Roughened HM multiwavelet filter

$$F_s^{-}(z) = D_{-2}^{-}z^{-2} + D_{-1}^{-}(s)z^{-1} + D_0^{-}(s) + D_1^{-}(s)z + D_2^{-}(s)z^2,$$

where the coefficients are given by:

$$D_{-2}^{-}(s) = \frac{1+2s}{12}\begin{bmatrix} -\sqrt{2} & \sqrt{2} \\ -2 & 2 \end{bmatrix},$$

$$D_{-1}^{-}(s) = \frac{1}{12}\begin{bmatrix} 6\sqrt{2} & (20s-8)\sqrt{2} \\ 12 & (20s-8)2 \end{bmatrix},$$

$$D_{0}^{-}(s) = \frac{1}{12}\begin{bmatrix} 0 & (4s-34)\sqrt{2} \\ 8s-20 & 0 \end{bmatrix},$$

$$D_{1}^{-}(s) = \frac{1}{12}\begin{bmatrix} -6\sqrt{2} & (20s-8)\sqrt{2} \\ 12 & 16-40s \end{bmatrix},$$

$$D_{2}^{-}(s) = \frac{1+2s}{12}\begin{bmatrix} \sqrt{2} & \sqrt{2} \\ -2 & -2 \end{bmatrix}. \tag{11}$$

Smoothened HM multiwavelet filter

$$F_{\tilde{s}}^{+}(z) = D_{-1}^{+}(\tilde{s})z^{-1} + D_0^{+}(\tilde{s}) + D_1^{+}(\tilde{s})z,$$

where the coefficients are given by:

$$D_{-1}^{+}(\tilde{s}) = \frac{1}{48}\begin{bmatrix} \frac{3\sqrt{2}}{2} & (4\tilde{s}-1)\frac{\sqrt{2}}{2} \\ 3 & (4\tilde{s}-1) \end{bmatrix} = \frac{1}{48}\begin{bmatrix} \frac{3\sqrt{2}}{2} & \frac{3(2+s)\sqrt{2}}{(5s-2)2} \\ 3 & \frac{3(2+s)}{(5s-2)} \end{bmatrix},$$

$$D_{0}^{+}(\tilde{s}) = \frac{1}{48}\begin{bmatrix} 0 & -3\sqrt{2} \\ -6 & 0 \end{bmatrix}, \tag{12}$$

$$D_{1}^{+}(\tilde{s}) = \frac{1}{48}\begin{bmatrix} \frac{-3\sqrt{2}}{2} & (4\tilde{s}-1)\frac{\sqrt{2}}{2} \\ 3 & -(4\tilde{s}-1) \end{bmatrix} = \frac{1}{48}\begin{bmatrix} \frac{-3\sqrt{2}}{2} & \frac{3(2+s)\sqrt{2}}{(5s-2)2} \\ 3 & \frac{-3(2+s)}{(5s-2)} \end{bmatrix},$$

see Figure 2.

4 Biorthogonal multiwavelets on the interval [0,1]

Hardin and Marasovich showed in [HM] that the truncation of their multiwavelets to the interval [0,1], plus minor adjustments at the boundary (those wavelets should be counted twice) produces certainly a Riesz basis for $L^2([0,1])$, but more than that holds: they are biorthogonal and they correspond to an MRA generated by the truncation of the multiscaling functions. In other words, in this case no work has to be done to produce boundary wavelets and scaling functions so that the MRA structure persists, see [HM, Sec 3.4, p.26].

The claim here is that we can truncate the new HM multiscaling functions and multiwavelets to the interval $[0,1]$, keep those that satisfy zero boundary values and get a biorthogonal basis for $H_0^1([0,1])$ by the same symmetry arguments used by Hardin and Marasovich. First we need to fix some ideas.

The Sobolev space $H^1(\mathbf{R}) = \{f \in L^2(\mathbf{R}) : Df = f' \in L^2(\mathbf{R})\}$. The derivative is to be understood in the distributional sense. This becomes a Hilbert space under the norm:

$$\|f\|^2_{H^1(\mathbf{R})} = \|f\|^2_{L^2(\mathbf{R})} + \|f'\|^2_{L^2(\mathbf{R})}.$$

Now it is well known that a function in $H^1(\mathbf{R})$ is equal almost everywhere to a locally absolutely continuous function, specifically, to an antiderivative of f'.

The space $H_0^1([0,1])$ can be thought as those absolutely continuous functions supported on $[0,1]$. More precisely:

$$H_0^1([0,1]) = clos_{H^1(\mathbf{R})} C_0^\infty([0,1]),$$

where $C_0^\infty([0,1])$ denotes the infinitely differentiable functions supported inside $[0,1]$.

4.1 HM biorthogonal multiwavelets on $L^2([0,1])$.

Hardin and Marasovich constructed a biorthogonal MRA on $[0,1]$ by truncating the HM multiscaling functions and multiwavelets. To be precise, let $\{V_j\}$ and $\{\tilde{V}_j\}$ be the MRAs generated by Φ_s and $\Phi_{\tilde{s}}$ respectively and let $\mathcal{V}_j = V_j\chi_{[0,1]}$ and $\tilde{\mathcal{V}}_j = \tilde{V}_j\chi_{[0,1]}$. Define $\mathcal{W}_j = \mathcal{V}_{j+1}\cap\tilde{\mathcal{V}}_j^\perp$ and $\tilde{\mathcal{W}}_j = \tilde{\mathcal{V}}_{j+1}\cap\mathcal{V}_j^\perp$ for $j\geq 0$; so by definition $\tilde{\mathcal{W}}_j\perp\mathcal{V}_j$ and $\mathcal{W}_j\perp\tilde{\mathcal{V}}_j$. For each $j\geq 0$ define:

$$\phi^i_{s,j,k}(x) = \begin{cases} 2^{\frac{1+j}{2}}\phi^i_s(2^jx-k)\chi_{[0,1]} & i=1,\ \ k=0,2^j \quad \text{(symmetric)} \\ 2^{\frac{j}{2}}\phi^i_s(2^jx-k) & i=2,\ \ k=0; \quad \text{(interior)} \\ & i=1,2,\ \ k=1,\ldots,2^j-1\,. \end{cases}$$

Similarly for $\tilde{\phi}^i_{s,j,k} = \phi^i_{\tilde{s},j,k}$. It can be shown that the collection $\{\phi^i_{s,j,k}, \tilde{\phi}^i_{s,j,k} : i=1,2,\ \ k=0,\ldots,2^j-1; \quad i=1,\ \ k=2^j\}$ forms a biorthogonal basis for $\mathcal{V}_j$ and $\tilde{\mathcal{V}}_j$. The only issue is what happens when pairing the "truly truncated" scaling functions with themselves. But in that case, by symmetry those that were orthogonal in the larger interval had to be orthogonal in the smaller one; and when paired with their duals, only half of the inner product will be accounted for, that is why we normalize by $\sqrt{2}$ in that case. Counting we see that $\dim\mathcal{V}_j = 2^{j+1}+1$.

By the biorthogonality condition $\mathcal{V}_j\cap\tilde{\mathcal{V}}_j^\perp = \{0\}$, hence $\mathcal{V}_{j+1} = \mathcal{V}_j\oplus\mathcal{W}_j$ and $\tilde{\mathcal{V}}_{j+1} = \tilde{\mathcal{V}}_j\oplus\tilde{\mathcal{W}}_j$. Therefore $\dim\mathcal{W}_j = 2^{j+1}$, and the 2^{j+1} functions:

$$\psi^i_{s,j,k}(x) = \begin{cases} 2^{\frac{1+j}{2}}\psi^i_s(2^jx-k)\chi_{[0,1]} & i=1,\ \ k=0,2^j \quad \text{(symmetric)} \\ 2^{\frac{j}{2}}\psi^i_s(2^jx-k) & i=1,2,\ \ k=1,\ldots,2^j-1 \ \ \text{(interior)}. \end{cases}$$

are in $\mathcal{V}_{j+1}$ and are orthogonal to $\tilde{\mathcal{V}}_j$, hence they are in $\mathcal{W}_j$. Together with corresponding $\tilde{\psi}^i_{s,j,k} = \psi^i_{\tilde{s},j,k}$ they generate a biorthogonal basis for $\mathcal{W}_j$ and $\tilde{\mathcal{W}}_j$. Moreover $L^2([0,1]) = \tilde{\mathcal{V}}_0\oplus(\oplus_{j\geq 0}\tilde{\mathcal{W}}_j)$.

See [HM, Sec 3.4, p. 26] for more details.

4.2 HM biorthogonal multiwavelets on $H_0^1([0,1])$.

This time we start with $\{V_j^-\}$ and $\{\tilde{V}_j^+\}$ the MRAs generated by Φ_s^- and $\Phi_{\tilde{s}}^+$ respectively. Let $\tilde{\mathcal{V}}_j^+$ be expanded by those truncates of $\Phi^+_{\tilde{s},j,k}$ to the interval $[0,1]$ with zero boundary values, that is the following functions:

$$\phi^{i,+}_{\tilde{s},j,k}(x)=\begin{cases} 2^{\frac{1+j}{2}}\phi^{i,+}_{\tilde{s}}(2^jx-k)\chi_{[0,1]} & i=2,\ k=0,2^j \quad \text{(antisymmetric)} \\ 2^{\frac{j}{2}}\phi^{i,+}_{\tilde{s}}(2^jx-k) & i=1,2,\ k=1,\ldots,2^j-1 \ \text{(interior)} \end{cases},$$

and let $\{\phi^{i,-}_{s,j,k}\}$ be the corresponding functions for $\mathcal{V}^-_j$. Let $\mathcal{W}^-_j=\mathcal{V}^-_{j+1}\cap(\tilde{\mathcal{V}}^+_j)^\perp$ and $\tilde{\mathcal{W}}^+_j=\tilde{\mathcal{V}}^+_{j+1}\cap(\mathcal{V}^-_j)^\perp$ for $j\geq 0$. The following 2^{j+1} functions are in $\tilde{\mathcal{V}}^+_{j+1}$:

$$\psi^{i,+}_{\tilde{s},j,k}(x)=\begin{cases} 2^{\frac{1+j}{2}}\psi^{i,+}_{\tilde{s}}(2^jx-k)\chi_{[0,1]} & i=1,\ k=0,2^j \quad \text{(antisymmetric)} \\ 2^{\frac{j}{2}}\psi^{i,+}_{\tilde{s}}(2^jx-k) & i=1,2,\ k=1,\ldots,2^j-1 \ \text{(interior)} \end{cases},$$

and let $\{\psi^{i,-}_{s,j,k}\}$ be the corresponding functions for $\mathcal{V}^-_{j+1}$.

The two systems are related by differentiation and integration, and inherit corresponding commutation relations.

Lemma 1 *The following hold:*

(a) *Differentiation and integration relations.*

$$D\Phi^+_{\tilde{s},j,k}=2^jT_{M^*}\Phi_{\tilde{s},j,k},\qquad D\Phi_{s,j,k}=-2^jT_M\Phi^-_{s,j,k},$$

$$D\Psi^+_{\tilde{s},j,k}=2^j\Psi_{\tilde{s},j,k}\qquad D\Psi_{s,j,k}=-2^j\Psi^-_{s,j,k},$$

where we keep only the components indicated by the index sets of the systems.

(b) *Commutation relations. Let $f\in H^1_0([0,1])$, then*

$$D\tilde{\mathcal{P}}^+_j+f=\tilde{\mathcal{P}}_jDf,\qquad D\tilde{\mathcal{Q}}^+_j+f=\tilde{\mathcal{Q}}_jDf;$$

where $\tilde{\mathcal{P}}^+_j$ and $\tilde{\mathcal{P}}_j$ denote the oblique projections onto the assymmetric, smoothened approximation space $\tilde{\mathcal{V}}^+_j$ and symmetric approximation space $\tilde{\mathcal{V}}_j$ on $[0,1]$ respectively; and $\tilde{\mathcal{Q}}^+_j$ and $\tilde{\mathcal{Q}}_j$ denote the oblique projections onto the corresponding detail spaces $\tilde{\mathcal{W}}^+_j$ and $\tilde{\mathcal{W}}_j$.

Proof: It should be clear what the notation means for the wavelets. For the detail spaces the differentiation relation is a consequence of the corresponding identity being true for the nontruncated wavelets. Notice that, among the indexed terms, for the smoothened wavelets whose nonrestricted supports overlap the boundary, we keep only the restrictions of the antisymmetric wavelets; and for the non-smoothened ones we just keep the restrictions of the symmetric terms. Since differentiation interchanges the two types of symmetry, the proof of the commutation relation for the projectors onto detail spaces $D\tilde{\mathcal{Q}}_j^+ = \tilde{\mathcal{Q}}_j D$ holds – up to a sign change – by the fundamental theorem of calculus (see the proof of Thm 4 below). For the scaling functions we postpone the proof to Appendix E. The proof of the commutation relations for the approximating spaces follows by an argument similar to that used for the proof of Theorem 2. $\square$

Theorem 4 *The families* $\{\phi^{i,+}_{\tilde{s},j,k}, \phi^{i,-}_{s,j,k} \,:\, i=2,\ \ k=0,2^j;\ i=1,2,\ k=1,\ldots,2^j-1\}$ *form a biorthogonal basis for* $\tilde{\mathcal{V}}_j^+$ *and* $\mathcal{V}_j^-$*. Each of the approximating spaces has dimension* 2^{j+1}*. The families* $\{\psi^{i,+}_{\tilde{s},j,k}, \psi^{i,-}_{s,j,k} \,:\, i=2,\ \ k=0,2^j;\quad i=1,2,\ k=1,\ldots,2^j-1\}$ *form a biorthogonal basis for* $\mathcal{W}_j^-$ *and* $\tilde{\mathcal{W}}_j^+$*.*

Together they form a biorthogonal MRA for $H_0^1([0,1])$*. Moreover*

$$H_0^1([0,1]) = \tilde{\mathcal{V}}_0^+ \oplus (\oplus_{j\geq 0}\tilde{\mathcal{W}}_j^+)$$

and

$$\|f\|_{H_0^1([0,1])} \sim \sum_{i,j,k}(1+2^{2j})|\langle f, \psi^{i,-}_{s,j,k}\rangle|^2 + \sum_{k=0,1} 2|\langle f, \phi^{2,-}_{s,0,k}\rangle|^2.$$

Proof: It is not hard to see that the wavelets at scale j lie in the $j+1^{\mathrm{th}}$ approximating space and are orthogonal to the dual of the j^{th} approximating space. The biorthogonality condition is quickly checked for both approximation and detail spaces, again the only trouble could come from pairing truly truncated wavelets and/or scaling functions with themselves. In that case we will be pairing two antisymmetric functions, whose product is an even function whose integral over the integral $[0,1]$ is half the integral over the interval $[-1,1]$. If the functions were orthogonal in the larger interval they would still be in the smaller one, otherwise we normalize multiplying by $\sqrt{2}$ so that when pairing a function with its dual we get 1. Therefore the dimension of $\mathcal{W}_j^-$ and $\tilde{\mathcal{W}}_j^+$ is 2^{j+1}. It only remains to be checked that the closure in $H_0^1([0,1])$ of the union of the $\tilde{\mathcal{V}}_j^+ = H_0^1([0,1])$. This will be a consequence of the commutation relations.

Let $\tilde{\mathcal{V}}^+ = \cup_{j\geq 0}\tilde{\mathcal{V}}_j^+$, and denote by $\tilde{\mathcal{P}}^+$ the oblique projection onto $\tilde{\mathcal{V}}^+$ parallel to $\tilde{\mathcal{V}}^- = \cup_{j\geq 0}\tilde{\mathcal{V}}_j^-$, that is $\tilde{\mathcal{V}}^+ = \tilde{\mathcal{V}}_0^+ \oplus (\oplus_{j\geq 0}\tilde{\mathcal{W}}_j^+)$ and for $f \in L^2([0,1])$:

$$\tilde{\mathcal{P}}^+ f = \sum_{i,j,k}\langle f, \psi^{i,-}_{s,j,k}\rangle \psi^{i,+}_{\tilde{s},j,k} + \sum_{k=0,1}\langle f, \phi^{2,-}_{s,0,k}\rangle \phi^{2,+}_{\tilde{s},0,k}.$$

We want to show that if $f \in H_0^1([0,1])$ then $\tilde{\mathcal{P}}^+ f = f$ and that the norm equivalence holds.

If $f \in H_0^1([0,1])$ then both $f, f' \in L^2([0,1])$. Moreover, we can assume that f is absolutely continuous. Since f' is square integrable we can expand it in the biorthogonal basis given by truncation of the HM multiwavelets. The following identities are justified at each scale because there they are finite. The sum over j then converges in L^2-norm:

$$\begin{aligned} f' = Df &= \sum_{i,j,k} \langle Df, \psi^i_{s,j,k}\rangle \psi^i_{\tilde{s},j,k} + P_0 Df \\ &= \sum_{i,j,k} \langle f, -D\psi^i_{s,j,k}\rangle \psi^i_{\tilde{s},j,k} + D\tilde{P}_0^+ f \\ &= \sum_{i,j,k} \langle f, 2^j \psi^{i,-}_{s,j,k}\rangle \psi^i_{\tilde{s},j,k} + D\sum_{k=0,1} \langle f, \phi^{2,-}_{s,0,k}\rangle \phi^{2,+}_{\tilde{s},0,k} \\ &= \sum_{i,j,k} \langle f, \psi^{i,-}_{s,j,k}\rangle 2^j \psi^i_{\tilde{s},j,k} + \sum_{k=0,1} \langle f, \phi^{2,-}_{s,0,k}\rangle D\phi^{2,+}_{\tilde{s},0,k}; \end{aligned}$$

therefore

$$\|f'\|_2^2 \sim \sum_{i,j,k} 2^{2j} |\langle f, \psi^{i,-}_{s,j,k}\rangle^2 + \sum_{k=0,1} |\langle f, \phi^{2,-}_{s,0,k}\rangle|^2.$$

We conclude that

$$Df = \sum_{i,j,k} \langle f, \psi^{i,-}_{s,j,k}\rangle D\psi^{i,+}_{\tilde{s},j,k} + D\tilde{P}_0^+ f$$

$$= D(\sum_{i,j,k} \langle f, \psi^{i,-}_{s,j,k}\rangle \psi^{i,+}_{\tilde{s},j,k}) + D\tilde{P}_0^+ f = D\tilde{\mathcal{Q}}^+ f.$$

The last identities hold at each scale, verifying the commutation relation $D\tilde{\mathcal{Q}}_j^+ = \tilde{\mathcal{Q}}_j D$ for each $j \geq 0$. Adding over the scales we get that $Df = \tilde{\mathcal{Q}} Df = D\tilde{\mathcal{Q}}^+ f$ as claimed. Integrating both sides we get that $f = \tilde{\mathcal{Q}}^+ f + C$, but by the absolute continuity of f and its boundary values we conclude that $C = 0$. Finally

$$\|f\|_2^2 \sim \sum_{i,j,k} |\langle f, \psi^{i,-}_{s,j,k}\rangle|^2 + \sum_{k=0,1} |\langle f, \phi^{2,-}_{s,0,k}\rangle|^2.$$

Coupled with the norm estimate for the derivative we conclude that

$$\begin{aligned} \|f\|_{H_0^1([0,1])} &= \|f\|_2^2 + \|f'\|_2^2 \sim \sum_{i,j,k} (1 + 2^{2j}) |\langle f, \psi^{i,-}_{s,j,k}\rangle|^2 \\ &\quad + \sum_{k=0,1} 2|\langle f, \phi^{2,-}_{s,0,k}\rangle|^2. \qquad \square \end{aligned}$$

5 Divergence-free biorthogonal multiwavelets

P. G. Lemarié-Rieusset introduced a technique for constructing biorthogonal pairs of compactly supported wavelets, such that one of the generators is divergence-free. Lemarié's construction is based on the existence of biorthogonal MRAs related by differentiation and integration (see [Le1]), which, as we just showed, applies to multiwavelets as well. We will quickly recall for the reader the steps in the construction of a biorthogonal basis of divergence-free multiwavelets for the particular HM multiwavelets discussed in the previous sections. We will restrict ourselves in this paper to dimension 2. The other interesting case for applications is dimension 3. The construction in all dimensions and for form-valued multiwavelets is presented in [LMP] when starting with an orthogonal MRA of multiplicity r. One can reproduce almost verbatim this result in the biorthogonal case, as long as the transition matrices can be chosen to be M and $-M^*$, as is the case in the modified Hardin-Marasovich MRA.

Let us first define the spaces in which we are interested.

The **flux spaces** considered are the "Sobolev" space:

$$H(div, \mathbf{R}^2) = \{\vec{F} \in (L^2(\mathbf{R}^2))^2 : \nabla \cdot \vec{F} \in L^2(\mathbf{R}^2)\},$$

and the subspace of divergence-free vector fields:

$$H^0(div, \mathbf{R}^2) = \{\vec{F} \in H(div, \mathbf{R}^2) : \nabla \cdot \vec{F} = 0\}.$$

These are Hilbert spaces under the norm:

$$\|\vec{F}\|^2_{H(div,\mathbf{R}^2)} = \|\vec{F}\|^2_{(L^2(\mathbf{R}^2))^2} + \|\nabla.\vec{F}\|^2_{L^2(\mathbf{R}^2)}.$$

Here the divergence operator $\nabla : H(div, \mathbf{R}^2) \to L^2(\mathbf{R}^2)$ is defined as usual by:

$$\nabla.\vec{F} = \frac{\partial F_1}{\partial x_1} + \frac{\partial F_2}{\partial x_2},$$

where the derivatives are understood in the distributional sense.

5.1 Lemarie's construction for HM multiwavelets

We will just record the generators at each step in the construction of a biorthogonal basis of compactly supported multiwavelets such that the reconstruction wavelets are divergence-free.

Step 1. Basis in $L^2(\mathbf{R})$ related by differentiation and integration.

In the previous sections we found a new MRA related by differentiation and integration to the modified Hardin-Marasovich MRA. Our *building blocks* are the multiscaling and multiwavelets corresponding to these particular biorthogonal MRAs, namely:

- Multiwavelets: $(\Psi_{\tilde{s}}, \Psi_s)$ and $(\Psi^+_{\tilde{s}}, \Psi^-_s)$,
- Multiscaling functions: $(\Phi_{\tilde{s}}, \Phi_s)$ and $(\Phi^+_{\tilde{s}}, \Phi^-_s)$.

Remember that the following identities hold:

$$D\Psi_s = -\Psi^-_s, \qquad D\Psi^+_{\tilde{s}} = \Psi_{\tilde{s}},$$

$$D\Phi_s = -T_M\Phi^-_s, \qquad D\Phi^+_{\tilde{s}} = T_{M^*}\Phi_{\tilde{s}}.$$

Step 2. Bases in $L^2(\mathbf{R}^2)$ - Tensor products.

We will construct bases in $L^2(\mathbf{R}^2)$ by the usual tensor product procedure which provides a multiresolution analysis for $L^2(\mathbf{R}^2)$, here the approximating spaces are $\mathcal{V}_j = V_j \otimes V'_j$, same holds for the dual spaces. The biorthogonal complements have 3 components:

$$\mathcal{W}_j = (W_j \otimes W'_j) \oplus (W_j \otimes V'_j) \oplus (V_j \otimes W'_j).$$

We are using different MRAs in each variable!

We will use the smoothened Hardin-Marasovich MRA $(\tilde{V}^+_j, \tilde{W}^+_j)$ in one variable, the modified Hardin-Marasovich MRA $(\tilde{V}_j, \tilde{W}_j)$ in the other variable for the synthesis or reconstruction wavelets. We will use the roughened Hardin-Marasovich MRA (V^-_j, W^-_j) in one variable, the dual of the modified Hardin-Marasovich MRA (V_j, W_j) in the other variable for the analysis wavelets.

There are four possible such biorthogonal bases, parametrized by $\epsilon \in E = \{(\epsilon_1, \epsilon_2) : \epsilon_k = 0, 1\}$. We will use the convention: $\Psi^1_{\tilde{s}} = \Psi^+_{\tilde{s}}$, $\Psi^0_{\tilde{s}} = \Psi_{\tilde{s}}$, and $\Psi^{-1}_s = \Psi^-_s$, $\Psi^0_s = \Psi_s$.

For each $\epsilon \in E$ the generators of such biorthogonal basis are:

$$\begin{aligned}
\Gamma^\epsilon_1(x,y) &= \Psi^{\epsilon_1}_{\tilde{s}}(x)^T \Psi^{\epsilon_2}_{\tilde{s}}(y), \\
\Gamma^\epsilon_2(x,y) &= \Psi^{\epsilon_1}_{\tilde{s}}(x)^T \Phi^{\epsilon_2}_{\tilde{s}}(y), \\
\Gamma^\epsilon_3(x,y) &= \Phi^{\epsilon_1}_{\tilde{s}}(x)^T \Psi^{\epsilon_2}_{\tilde{s}}(y).
\end{aligned}$$

Notice that each generator is a 2×2 matrix, each entry of which encodes a basis element. Hence we are talking about 12 generators at this level.

The duals are given by $\tilde{\Gamma_k^\epsilon} = \Gamma_k^{-\epsilon}$, for $k = 1, 2, 3$.

Step 3. Bases in $(L^2(\mathbf{R}^2))^2$ - Componentwise.

Consider the standard basis in $\mathbf{R}^2$: $\vec{e}_1 = (1,0)^T$, $\vec{e}_2 = (0,1)^T$.

For each pair of indexes $\epsilon, \epsilon' \in E$ we can construct a biorthogonal basis in $(L^2(\mathbf{R}^2))^2$ by assigning the basis Γ_k^ϵ to the first component, and the basis $\Gamma_k^{\epsilon'}$ to the second component. In other words, a biorthogonal basis in $(L^2(\mathbf{R}^2))^2$ will be generated by $\{\Gamma_k^\epsilon \vec{e}_1, \Gamma_k^{\epsilon'} \vec{e}_2\}$, where the notation $\Gamma\vec{e}$ is a 2×2 matrix whose entries are now the vector fields $\Gamma_{i,j}\vec{e}$.

We will choose: $\epsilon = (1,0)$ and $\epsilon' = (0,1)$. And we will denote the generators of this basis by:

$$\vec{\Gamma}_k^{(1,0)} = \Gamma_k^\epsilon \vec{e}_1; \qquad \vec{\Gamma}_k^{(0,1)} = \Gamma_k^{\epsilon'} \vec{e}_2.$$

The dual wavelets are given by taking duals in each component:

$(\tilde{\vec{\Gamma}})_k^\epsilon = \vec{\Gamma}_k^{-\epsilon}$.

We are talking now about 24 generators, and 24 dual generators at this level.

We will list here the 6 "matrix generators" (each of them encodes 4 vector fields). The duals are found by replacing plus by minus, and $\tilde{s}$ by s. In the first three we integrate the first variable of the first component:.

$$\begin{aligned}
\vec{\Gamma}_1^{(1,0)}(x,y) &= (\Psi_{\tilde{s}}^{+}(x)^T \Psi_{\tilde{s}}(y), 0), \\
\vec{\Gamma}_2^{(1,0)}(x,y) &= (\Psi_{\tilde{s}}^{+}(x)^T \Phi_{\tilde{s}}(y), 0), \\
\vec{\Gamma}_3^{(1,0)}(x,y) &= (\Phi_{\tilde{s}}^{+}(x)^T \Psi_{\tilde{s}}(y), 0);
\end{aligned}$$

in the second three we integrate the second variable of the second component:

$$\begin{aligned}
\vec{\Gamma}_1^{(0,1)}(x,y) &= (0, \Psi_{\tilde{s}}(x)^T \Psi_{\tilde{s}}^{+}(y)), \\
\vec{\Gamma}_2^{(1,0)}(x,y) &= (0, \Psi_{\tilde{s}}(x)^T \Phi_{\tilde{s}}^{+}(y)), \\
\vec{\Gamma}_3^{(1,0)}(x,y) &= (0, \Phi_{\tilde{s}}(x)^T \Psi_{\tilde{s}}^{+}(y)).
\end{aligned}$$

Step 4. Basis in $H^0(div, \mathbf{R}^2)$.

We are going to list now 12 linear combinations of the vector fields listed above and their integer translates. This time we will also list the 12 dual generators, since they are not exactly what one would expect. It can be shown that these will be the generators of a complete biorthogonal basis in $H^0(div, \mathbf{R}^2)$, see [Le1], [LMP].

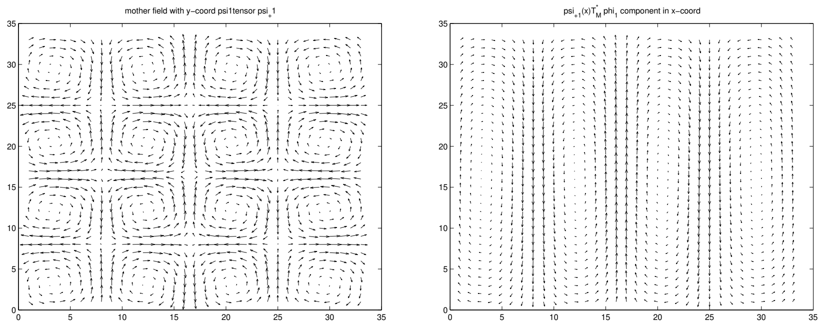

Figure 3. Plots of the upper left components of $\vec{\Psi}_1$ and $\vec{\Psi}_2$.

Again using our compact notation: $(A, B) = \{(a_{ij}, b_{ij})\}_{i,j=1,2}$, each generator encodes 4 vector fields. Here they are:

$$\begin{aligned}
\vec{\Psi}_1(x,y) &= (\Psi_{\tilde{s}}^+(x)^T \Psi_{\tilde{s}}(y), -\Psi_{\tilde{s}}(x)^T \Psi_{\tilde{s}}^+(y)), \\
\vec{\Psi}_2(x,y) &= (-\Psi_{\tilde{s}}^+(x)^T T_M^* \Phi_{\tilde{s}}(y), \Psi_{\tilde{s}}(x)^T \Phi_{\tilde{s}}^+(y)), \\
\vec{\Psi}_3(x,y) &= (\Phi_{\tilde{s}}^+(x)^T \Psi_{\tilde{s}}(y), -T_M^* \Phi_{\tilde{s}}(x)^T \Psi_{\tilde{s}}^+(y)).
\end{aligned}$$

These are clearly divergence-free vector fields, as can be seen when differentiating and using the differentiation properties of the HM smoothened multiscaling and multiwavelets. There are 12 generators at this level. The corresponding duals are:

$$\begin{aligned}
\tilde{\vec{\Psi}}_1(x,y) &= (\tilde{\vec{\Gamma}})_1^{(-1,0)}(x,y) = (\Psi_s^-(x)^T \Psi_s(y), 0) \\
\tilde{\vec{\Psi}}_2(x,y) &= (\tilde{\vec{\Gamma}})_2^{(0,-1)}(x,y) = (0, \Psi_s(x)^T \Phi_s^-(y)) \\
\tilde{\vec{\Psi}}_3(x,y) &= (\tilde{\vec{\Gamma}})_3^{(-1,0)}(x,y) = (\Phi_s^-(x)^T \Psi_s(y), 0).
\end{aligned}$$

The dilates and translates $\{\vec{\Psi}_{i,\lambda}\}$ form a Riesz basis for $H^0(div, \mathbf{R}^2)$; the dual basis characterizes the dual space of $H^0(div, \mathbf{R}^2)$. See the components of some of the divergence-free wavelets corresponding to $s = 0$ and $\tilde{s} = -1/2$ in Figure 3. The supports in those figures are not normalized.

Note: If we were doing this in dimension 3, at this last step we will be dealing with 112 generators, and 112 duals. It is not clear at all that this is the way to go, despite the advantages in defining boundary wavelets, as will be explained in the next section. Of course the dimensionality curse will worsen if we increase the multiplicity of the MRA.

5.2 HM Divergence-free multiwavelets on $[0,1]^2$

Inspired by Lemarié's construction of divergence-free compactly supported wavelets, Urban identified those properties of a wavelet system on $\Omega \subset \mathbf{R}^2$, some open bounded domain with Lipschitz-continuous boundary, that are needed to proceed with the subsequent construction of wavelet bases for the appropiate flux space, see [U, p.6].

We will restrict ourselves to the case $\Omega = [0,1]^2$ The **flux spaces** considered now are,

$$H(div,[0,1]^2) := \{\vec{F} \in (L^2([0,1]^2))^2 : \nabla \cdot \vec{F} \in L^2([0,1]^2)\},$$

and the subspace of divergence-free vector fields:

$$H^0(div,[0,1]^2) := \{\vec{F} \in H(div,[0,1]^2) : \nabla \cdot \vec{F} = 0\}.$$

These are Hilbert spaces under the norm:

$$\|\vec{F}\|^2_{H(div,[0,1]^2)} := \|\vec{F}\|^2_{(L^2([0,1]^2))^2} + \|\nabla.\vec{F}\|^2_{L^2([0,1]^2)},$$

and we define

$$\begin{aligned} H_p(div,[0,1]^2) := clos_{H(div,[0,1]^2)}\{\vec{F} \in (C^\infty([0,1]^2))^2 :& \\ F_1(0,y) = F_1(1,y) = 0, \ \ F_2(x,0) = F_2(x,1) = 0\}&; \end{aligned}$$

and

$$H^0_p(div,[0,1]^2) := H^0(div,[0,1]^2) \cap H_p(div,[0,1]^2).$$

The assumption formulated by Urban for the 2 dimensional case adapted to tangential boundary values would be:

Assumption 1 [Urban]: *For all $\epsilon = (\epsilon_1, \epsilon_2) \in E$ and for all $i \in 0,1$ such that $\epsilon - \delta_i \in E$, there exist biorthogonal systems $\{\Gamma^\epsilon_\lambda, \tilde{\Gamma}^\epsilon_\lambda\}_{\lambda \in \Lambda}$ in $L^2([0,1]^2)$ such that:*

- $\partial_i \Gamma^\epsilon = D^i \Gamma^{\epsilon - \delta_i}$ *where Γ^ϵ is viewed as a a column vector and D^i are sparse (only a fixed and small number of entries are non-zero), regular transformation.*
- *The vector field $(\Gamma^{(1,0)}_\lambda, \Gamma^{(0,1)}_\lambda)$ is in $H_p(div,[0,1]^2)$.*
- *If an entry $D^i_{\lambda,\mu} \neq 0$ then $|\lambda| = |\mu|$.*
- *The matrix $D^i_j = \{D^i_{\lambda,\mu}\}_{|\lambda|=|\mu|=j}$ is invertible for all j.*

This assumption will imply that certain commutation relations hold between the biorthogonal projectors and partial derivatives.

In the construction of the divergence-free multiwavelets we used as building blocks the smoothed out wavelets and the HM wavelet. We want to use their truncations as building blocks, hence the divergence-free multiwavelets can be thought as elements of:

$$\big(H_0^1([0,1])\otimes L^2([0,1)]), L^2([0,1)])\otimes H_0^1([0,1))\big)\cap H_0^0(div,[0,1]^2).$$

Consider the tensor product bases defined in Section 5, except that here we are using the truncated systems as described in section 4. Proceeding like in the $\mathbf{R}^2$ case we obtain biorthogonal systems that satisfy Urban's assumption. Hence we obtain the desired basis. Notice that there is a slight variation in the boundary conditions considered: the vector fields we are considering are tangential to the boundary of the unit square. This is a reasonable boundary condition for divergence-free velocity fields. Actually, the wavelets presented here are ideally adapted to such boundaries. More details will appear in forthcoming work.

6 Appendix A: Fractal interpolation functions

Fractal interpolation functions and Lipschitz-α continuity of HM multiscaling functions. The construction of the multiscaling functions starts with the hat function $h(x) = (1-|x|)\chi_{[-1,1]}(x)$, which is a scaling function with refinement equation: $h(x) = \frac{1}{2}h(2x+1) + h(2x) + \frac{1}{2}h(2x-1)$ for all $x \in R$, see [HM, Sec. 3.1 p.15].

Let $C_0^\alpha([0,1])$ denote the space of Lipschitz α continuous functions on $\mathbf{R}$ supported in $[0,1]$, for $0 \le \alpha \le 1$. This space consists of those continuous funtions in $\mathbf{R}$ supported on $[0,1]$ such $|f(x)-f(y)| \le C|x-y|^\alpha$. It is a Banach space with the norm

$$\|f\|_\alpha = \sup_{x,y\in[0,1]} \frac{|f(x)-f(y)|}{|x-y|^\alpha},$$

when $0 < \alpha \le 1$, and when $\alpha = 0$ we use the uniform norm. Note that $h \in C_0^\alpha([0,1])$ for all $0 \le \alpha \le 1$.

We seek a function $\omega \in C_0^\alpha([0,1])$, which satisfies the refinement equation:

$$\omega(x) = h(2x-1) + s\omega(2x) + s\omega(2x-1), \qquad x \in \mathbf{R},$$

for a constant s which is the parameter s in the filters discussed in this paper. The normalization implies that $\omega(1/2) = 1$. Such a function is called a fractal interpolation function.

It is not necessary to have the same coefficient s for both $\omega(2x)$ and $\omega(2x-1)$; different cases were analyzed by GHM, cf., [HM]. Choosing the same s imposes symmetry about $1/2$. It would be possible to start the process with a higher order spline as well.

Consider the linear operator defined on continuous functions in $C_0([0,1])$ by:

$$\Gamma_s f(x) = sf(2x) + sf(2x-1).$$

Notice that the closed subspace $S^\alpha_{1/2}$ of symmetric functions about $1/2$ is invariant under Γ_s.

Lemma 2 *Let $0 \leq \alpha \leq 1$. If $|s| < 2^{-\alpha}$ then Γ_s is a contraction on $S^\alpha_{1/2}$ the closed subspace of $C^\alpha_0([0,1])$. Therefore there is a unique fixed point $\omega \in S^\alpha_{1/2}$ that satisfies the refinement equation:*

$$\omega(x) = h(2x-1) + s\omega(2x) + s\omega(2x-1).$$

Proof: If $x, y \in [0, 1/2]$ or $x, y \in [1/2, 0]$, $f \in S^\alpha_{1/2}$ then

$$\begin{aligned}
\frac{|\Gamma_s f(x) - \Gamma_s f(y)|}{|x-y|^\alpha} &\leq s\left(\frac{|f(2x) - f(2y)|}{|x-y|^\alpha} + \frac{|f(2x-1) - f(2y-1)|}{|x-y|^\alpha}\right) \\
&\leq 2^\alpha s \|f\|_\alpha,
\end{aligned}$$

the last inequality because one summand in the middle expression always vanishes.

Suppose that $x \in [0, 1/2]$ and $y \in [1/2, 0]$ (of course we could interchange their roles) then by the symmetry about $1/2$, $f(2x) = f(1-2x)$, hence

$$\begin{aligned}
\frac{|\Gamma_s f(x) - \Gamma_s f(y)|}{|x-y|^\alpha} &= \frac{s|f(2x) - f(2y-1)|}{|x-y|^\alpha} \\
&= \frac{s|f(1-2x) - f(2y-1)|}{|1-2x-(2y-1)|^\alpha} \frac{|1-2x-(2y-1)|^\alpha}{|x-y|^\alpha} \\
&\leq 2^\alpha s \|f\|_\alpha,
\end{aligned}$$

the last inequality because $|1-x-y| \leq |\frac{1}{2} - x| + |y - \frac{1}{2}| = |x-y|$. Hence $\|\Gamma_s\|_{op} \leq 2^\alpha s < 1$, and Γ_s is a contraction in $S^\alpha_{1/2}$ as claimed. One actually gets a contraction in $C_0([0,1])$ for all $|s| < 1$ as was shown in [HM].

We want to find a solution in $S^\alpha_{1/2}$ for the equation:

$$(I - \Gamma_s)\omega = g, \qquad g(x) = h(2x-1).$$

Since Γ_s is a contraction and the function g is Lipschitz continuous, supported on $[0,1]$ and symmetric about $1/2$, it belongs to $S^\alpha_{1/2}$ for all $0 \leq \alpha \leq$

1, then by a fixed point argument we know there is a unique solution to the above equation, which is given by the Neumann series: $\omega = \sum_{n=0}^{\infty} \Gamma_s^n g$.

□

The scaling functions are found biorthogonalizing the pairs (h, ω_s) and $(h, \omega_{\tilde{s}})$, so that we respect the MRA structure. In the biorthogonalization process one observes that $\tilde{s} = \frac{1+2s}{5s-2}$, see [HM].

7 Appendix B: Modifications to the HM filters

Once they had the fractal interpolation functions, Hardin and Marasovich introduce a few parameters in defining the biorthogonal scaling vectors

$$\phi_s^1 = \gamma(h - \alpha\omega_s - \beta\omega_s(\cdot + 1)), \qquad \phi_s^2 = \delta\omega_s,$$
$$\phi_{\tilde{s}}^1 = \tilde{\gamma}(h - \tilde{\alpha}\omega_{\tilde{s}} - \tilde{\beta}\omega_{\tilde{s}}(\cdot + 1)), \qquad \phi_{\tilde{s}}^2 = \delta_{\tilde{s}}\omega_{\tilde{s}}.$$

The parameters $\delta, \gamma, \alpha, \beta$ are all functions of s; $\tilde{\delta}, \tilde{\gamma}, \tilde{\alpha}, \tilde{\beta}$ are functions of $\tilde{s}$, and $\tilde{s} = \frac{1+2s}{5s-2}$, see [HM]. Obviously, except in the degenerate case, these relations define refinable functions that generate the same MRAs as (h, ω_s) and $(h, \omega_{\tilde{s}})$. The parameters are chosen with the help of Gram-Schmidt to ensure biorthogonality.

To ensure symmetry of the scaling functions Hardin and Marasovich set $\alpha = \beta$ and $\tilde{\alpha} = \tilde{\beta}$. They then set $\delta = \tilde{\delta}$ and $\gamma = \tilde{\gamma}$. It turns out that γ does not depend on s but δ does. In fact, in the HM setup it depends nonlinearly on s. *The modification we made, in effect, chooses δ to depend linearly on s.* This is the only modification that we made to the HM parameterization. One then defines the entries of the filter H_s as linear functions of the parameters $\gamma, \alpha, \delta, s$, which guarantees that $\tilde{H}_s = H_{\tilde{s}}$). Similarly for the wavelet filters F_s and $\tilde{F}_s$, our choice ensures $\tilde{F}_s = F_{\tilde{s}}$).

8 Appendix C: Conditions of biorthogonality

Here we show how to check that the initial filters satisfy the biorthogonality conditions:

$$\begin{aligned}
H_s(z)H_{\tilde{s}}^*(z) + H_s(-z)H_{\tilde{s}}^*(-z) &= I \\
F_s(z)F_{\tilde{s}}^*(z) + F_s(-z)F_{\tilde{s}}^*(-z) &= I \\
H_s(z)F_{\tilde{s}}^*(z) + H_s(-z)F_{\tilde{s}}^*(-z) &= 0 \\
F_s(z)H_{\tilde{s}}^*(z) + F_s(-z)H_{\tilde{s}}^*(-z) &= 0
\end{aligned}$$

We will only consider the first equation, the others being similar. For the identity on the low pass filters, one has

$$H_s(z) = \begin{bmatrix} \frac{12z^2+(z+z^3)(-2+8s)}{24z^2} & \frac{-(1+2s)\sqrt{2}(1+z^3)+(z+z^2)(5-2s)\sqrt{2}}{24z^2} \\ \frac{8z^3\sqrt{2}(1-s)}{24z^2} & \frac{(z^2+z^3)(8+4s)}{24z^2} \end{bmatrix}$$

$$H_{\tilde{s}}^*(z) = \begin{bmatrix} \frac{6z\left(s+2+\frac{10}{z}s-\frac{4}{z}+\frac{1}{z^2}s+\frac{2}{z^2}\right)}{24(5s-2)} & \frac{z^2(\frac{24}{z^3}\sqrt{2}(s-1))}{24(5s-2)} \\ \frac{-3z^2\sqrt{2}\left(\frac{3}{z^2}s-\frac{10}{z}s+\frac{4}{z}+3s\right)\left(1+\frac{1}{z}\right)}{24(5s-2)} & \frac{12(4s-1)\left(1+\frac{1}{z}\right)}{24(5s-2)} \end{bmatrix}$$

Writing $A(z) = H_s(z)H_{\tilde{s}}^*(z)$, the entries of A are:

$$\begin{aligned} a_{11} &= \frac{1}{96(5s-2)\,z^3}\,(3s - 96z^3 + 16z^2 - 11z^2 s + 240z^3 s + 6s^2 \\ &\qquad +58z^2s^2 - 11z^4 s + 58z^4s^2 + 3z^6 s + 6z^6s^2 + 16z^4) \\ a_{12} &= -\frac{1}{48(5s-2)\,z^3}\sqrt{2}\,(-68z^2 s + 34z^2 + 2s - 1 + 2z^4 s - z^4 \\ &\qquad +8s^2 + 8z^4s^2 + 16z^2s^2 \\ a_{21} &= -\frac{1}{48z}\sqrt{2}\frac{26z^2s^2 - 76z^2 s + 32z^2 + 6s + 3s^2 + 6z^4 s + 3z^4s^2}{5s-2} \\ a_{22} &= \frac{1}{12z}\frac{30zs - 12z + 7s - 2 + 4s^2 + 7z^2 s - 2z^2 + 4z^2s^2}{5s-2}. \end{aligned}$$

One sees, therefore, that all of the terms in a_{11} and a_{12} are odd except for the constant terms which equal $1/2$. On the other hand, all of the terms of a_{12} and a_{21} are odd. So it is clear that $a_{11}(z)+a_{11}(-z) = a_{22}(z)+a_{22}(-z) = 1$ while $a_{12}(z) + a_{12}(-z) = 0 = a_{21}(z) + a_{21}(-z)$. This verifies the first of the conditions of biorthogonality. As mentioned, the others are similar.

The conditions of biorthogonality for the smoothened and roughened filters are:

$$\begin{aligned} H_s^-(z)(H_{\tilde{s}}^+)^*(z) + H_s^-(-z)(H_{\tilde{s}}^+)^*(-z) &= cI \\ F_s^-(z)(F_{\tilde{s}}^+)^*(z) + F_s^-(-z)(F_{\tilde{s}}^+)^*(-z) &= cI \\ H_s^-(z)(F_{\tilde{s}}^+)^*(z) + H_s^-(-z)(F_{\tilde{s}}^+)^*(-z) &= 0 \\ F_s^-(z)(H_{\tilde{s}}^+)^*(z) + F_s^-(-z)(H_{\tilde{s}}^+)^*(-z) &= 0\,. \end{aligned}$$

Once one has verified the conditions for the starting filters, the conditions for the smoothened and roughened filters are simple. For example, one has:

$$\begin{aligned}
H_s^-(z)(H_{\tilde{s}}^+)^*(z) + H_s^-(-z)(H_{\tilde{s}}^+)^*(-z) &= \\
2M^{-1}(z^2)H_s(z)M(z)[\frac{1}{2}M^*(z^2)H_{\tilde{s}}(z)(M^*(z))^{-1}] & \\
+2M^{-1}((-z)^2)H_s(-z)M(-z)[\frac{1}{2}M^*((-z)^2)H_{\tilde{s}}(-z)(M^*(-z))^{-1}] &= \\
M^{-1}(z^2)H_s(z)H_{\tilde{s}}^*(z)M^*(z^2) + M^{-1}(z^2)H_s(-z)H_{\tilde{s}}^*(-z)M^*(z^2) &= I
\end{aligned}$$

The computations for the other conditions are similar. In more explicit terms, the entries of the matrix $(H_{\tilde{s}}^+)^*(z)H_s^-(z)$ are:

$$\begin{aligned}
a_{11} &= -\frac{1}{96(5s-2)z^3}(-240sz^3 + 96z^3 + 2s - z^6 + 2z^6s - 1 - 122sz^4 \\
&\quad -8s^2z^4 - 8z^2s^2 + 49z^2 + 8s^2 + 8z^6s^2 - 122z^2s + 49z^4) \\
a_{12} &= -\frac{1}{96(5s-2)z^3}(-2s - z^6 + 2z^6s - 70sz^4 + 8s^2z^4 \\
&\quad -8z^2s^2 - 35z^2 - 8s^2 + 8z^6s^2 + 70z^2s + 1 + 35z^4) \\
a_{21} &= \frac{1}{96(5s-2)z^3}(-5s - z^6 + 5z^6s + 1 - 171sz^4 + 54s^2z^4 \\
&\quad -54z^2s^2 - 63z^2 - 14s^2 + 14z^6s^2 + 171z^2s + 63z^4) \\
a_{22} &= \frac{1}{96(5s-2)z^3}(240sz^3 - 96z^3 + 5s - z^6 + 5z^6s - 77sz^4 + 82s^2z^4 \\
&\quad +82z^2s^2 + 49z^2 + 14s^2 + 14z^6s^2 - 77z^2s - 1 + 49z^4)
\end{aligned}$$

and, as before, all of the odd terms will cancel when adding the values at $-z$ which gives the desired identity for the smoothened and roughened low-pass filters.

9 Appendix D: Symmetry and support properties

Here we verify parts (b) and (c) of Theorem 2 – that symmetry and support properties are preserved under smoothening and roughening the HM scaling functions.

Theorem 2(b) *Both components of Φ_s^- are piecewise continuous and supported on $[-1,1]$; the first component $\Phi_s^{1,-}$ is symmetric about 0, the sec-*

ond $\Phi_s^{2,-}$ is antisymmetric about 0. Moreover, let $\alpha = \Phi_s^{1,-}\chi_{[-1,0]}$ and $\beta = \Phi_s^{1,-}\chi_{[0,1]}$, then $\Phi_s^{1,-} = \alpha + \beta$ and $\Phi_s^{2,-} = \alpha - \beta$.

Proof of (b): We mentioned that $M(z) = E(z)M(z^{-1})E_-^{-1}(z)$, where

$$E(z) = \begin{pmatrix} -1 & 0 \\ 0 & -z \end{pmatrix}, \qquad E_-(z) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

This immediately gives the right symmetries by Strela's symmetry criterion in p.13. That is, the first component $\phi_s^{1,-}$ is symmetric about 0, the second $\phi_s^{2,-}$ is antisymmetric about 0. In principle the supports could be larger than $[-1,1]$. From the relation $D\Phi_s = T_M\Phi_s^-$, and the fact that with the choice of transition matrix M we have:

$$\begin{aligned} T_M\Phi_s^- &= \begin{pmatrix} 0 & 2\sqrt{2}I \\ I-S & -I-S \end{pmatrix}\begin{pmatrix} \phi_s^{1,-} \\ \phi_s^{2,-} \end{pmatrix} \\ &= \begin{pmatrix} 2\sqrt{2}\phi_s^{2,-} \\ (I-S)\phi_s^{1,-} - (I+S)\phi_s^{2,-} \end{pmatrix}; \end{aligned}$$

we get $-D\phi_s^1 = 2\sqrt{2}\phi_s^{2,-}$ and $-D\phi_s^2 = (I-S)\phi_s^{1,-} - (I+S)\phi_s^{2,-}$.

The first identity implies that $\phi_s^{2,-}$ has same support and symmetry as $D\phi_s^1$, namely is antisymmetric about 0 and is supported on $[-1,1]$.

The second identity implies that the right hand side must be supported on $[0,1]$ and must be antisymmetric about $1/2$ as $D\phi_s^2$ is.

Let $\phi_s^{2,-} = \alpha_1 - \beta_1$, where α_1 is supported on $[-1,0]$ and β_1 is supported on $[0,1]$, and $\alpha_1(x) = \beta_1(-x)$ (by the antisymmetry of $\phi_s^{2,-}$). Recall that $Sf(x) = f(x-1)$, hence the support of the shifted function is shifted one unit to the right.

By the symmetry analysis $\phi_s^{1,-}$ is symmetric about 0. Let $\phi_s^{1,-} = \alpha_2 + \beta_2 + \eta$, where α_2 is supported on $[-1,0]$, β_2 is supported on $[0,1]$ and η is supported outside $[-1,1]$; necessarily $\alpha_2(x) = \beta_2(-x)$ and $\eta(x) = \eta(-x)$.

With this notation we conclude that

$$-D\phi_s^2 = (\alpha_2 + \beta_2 + \eta) - S(\alpha_2 + \beta_2 + \eta) - (\alpha_1 - \beta_1) - S(\alpha_1 - \beta_1).$$

By support considerations the only survivors on the right hand side must be $(\beta_2 + \beta_1) - S(\alpha_2 + \alpha_1)$, everything else must cancel, hence:

$$\alpha_2 - \alpha_1 = S\eta\chi_{[-1,0]},$$

$$S(\beta_2 - \beta_1) = \eta\chi_{[1,2]},$$

and also $\eta\chi_{[-1,2]^c} = S\eta\chi_{[-1,2]^c}$, which implies $\eta(x) = \eta(x+1)$ for all $x \in [-1,2]^c$, since $\eta \in L^2(\mathbf{R})$ then $\eta\chi_{[-1,1]^c} = 0$, which in turn implies that

$\eta \equiv 0$. Therefore $\alpha_2 = \alpha_1 = \alpha$ and $\beta_1 = \beta_2 = \beta$. Finally we conclude that: $\phi_s^{1,-} = \alpha + \beta$ and $\phi_s^{2,-} = \alpha - \beta$, which was to be proved. $\square$.

Note that the relationship in terms of α, β obviously holds for the case $s = 0$ in Figure 2.

Theorem 2(c) *Both components of $\Phi_{\tilde{s}}^+$ are supported on $[-1, 1]$. The first component $\phi_{\tilde{s}}^{1,+}$ is symmetric about 0. The second component $\phi_{\tilde{s}}^{2,+}$ is antisymmetric about 0.*

Proof of (c): This time $D\Phi_{\tilde{s}}^+ = T_{M^*}\Phi_{\tilde{s}}$ where

$$\begin{aligned} D\Phi_{\tilde{s}}^+ &= T_M^*\Phi_{\tilde{s}} = \begin{pmatrix} 0 & I - S^{-1} \\ 2\sqrt{2}I & -I - S^{-1} \end{pmatrix} \begin{pmatrix} \phi_{\tilde{s}}^1 \\ \phi_{\tilde{s}}^2 \end{pmatrix} \\ &= \begin{pmatrix} (I - S^{-1})\phi_{\tilde{s}}^2 \\ 2\sqrt{2}\phi_{\tilde{s}}^1 - (I + S^{-1})\phi_{\tilde{s}}^2 \end{pmatrix}. \end{aligned}$$

Notice that $S^{-1}f(x) = f(x+1)$, therefore the support of $S^{-1}f$ is shifted one unit to the left. In particular: $D\phi_{\tilde{s}}^{1,+} = (I - S^{-1})\phi_{\tilde{s}}^2$ is supported on $[-1, 1]$, is antisymmetric about 0, and on the half interval $[0, 1]$ coincides with $\phi_{\tilde{s}}^2$, and hence is symmetric about $1/2$; therefore its antiderivative: $\phi_{\tilde{s}}^{1,+}$ is supported on $[-1, 1]$, symmetric about 0.

Similarly from the support and symmetry properties of $\Phi_{\tilde{s}}$ we conclude that $D\phi_{\tilde{s}}^{2,+} = 2\sqrt{2}\phi_{\tilde{s}}^1 - (I + S^{-1})\phi_{\tilde{s}}^2$ is supported on $[-1, 1]$ and is symmetric about 0. Therefore its antiderivative: $\phi_{\tilde{s}}^{2,+}$ is supported on $[-1, 1]$, and is antisymmetric about 0, as claimed. $\square$

10 Appendix E

Proof of Lemma 1 (a)

By

$$D\Phi_{\tilde{s},j,k}^+ = 2^j T_{M^*}\Phi_{\tilde{s},j,k}, \qquad D\Phi_{s,j,k} = -2^j T_M \Phi_{s,j,k}^-,$$

we mean more precisely,

$$D\Phi_{\tilde{s},j,k}^+(x) = \begin{pmatrix} D\phi_{\tilde{s},j,k}^{1,+}(x) \\ D\phi_{\tilde{s},j,k}^{2,+}(x) \end{pmatrix} = 2^j \begin{pmatrix} 2^{-j/2} D\phi_{\tilde{s}}^{1,+}(2^j x - k)\chi_{[0,1]} \\ c_k 2^{-j/2} D\phi_{\tilde{s}}^{2,+}(2^j x - k)\chi_{[0,1]} \end{pmatrix} =$$

$$= 2^j \begin{pmatrix} 2^{-j/2}(I - S^{-1})\phi_{\tilde{s}}^2(2^j x - k)\chi_{[0,1]} \\ c_k 2^{-j/2}\big(2\sqrt{2}\phi_{\tilde{s}}^1(2^j x - k) - (I + S^{-1})\phi_{\tilde{s}}^2(2^j x - k)\big)\chi_{[0,1]} \end{pmatrix},$$

where $c_k = \sqrt{2}$ if $k = 0, 2^j$, and one otherwise; we are using the fact that $D\Phi_{\tilde{s}}^{+} = T_{M^*}\Phi_{\tilde{s}}$, and the definition of the transformation T_{M^*} (see Appendix D).

For the interior scaling functions, that is when $i = 1, 2;\ \ k = 1, \ldots 2^j - 1$ we get:

$$\begin{pmatrix} D\phi^{1,+}_{\tilde{s},j,k} \\ D\phi^{2,+}_{\tilde{s},j,k} \end{pmatrix} = 2^j \begin{pmatrix} \phi^2_{\tilde{s},j,k} - \phi^2_{\tilde{s},j,k-1} \\ 2\sqrt{2}\phi^1_{\tilde{s},j,k} - \phi^2_{\tilde{s},j,k} - \phi^2_{\tilde{s},j,k-1} \end{pmatrix}.$$

And for the truly truncated ones, when $i = 2$ and $k = 0, 2^j$, then:

$$D\phi^{2,+}_{\tilde{s},j,0} = 2^j\sqrt{2}(2\phi^1_{\tilde{s},j,0} - \phi^2_{\tilde{s},j,0}),$$

$$D\phi^{2,+}_{\tilde{s},j,2^j} = 2^j\sqrt{2}(2\phi^1_{\tilde{s},j,2^j} - \phi^2_{\tilde{s},j,2^j-1}).$$

For the dual scaling functions we get something similar, using this time that $D\Phi_s = -T_M\Phi_s^{-}$, and the definition of the transformation T_M (see Appendix D),

$$D\Phi_{s,j,k}(x) = \begin{pmatrix} D\phi^1_{s,j,k}(x) \\ D\phi^2_{s,j,k}(x) \end{pmatrix} = 2^j \begin{pmatrix} c_k 2^{-j/2} D\phi^1_s(2^jx-k)\chi_{[0,1]} \\ 2^{-j/2} D\phi^2_s(2^jx-k)\chi_{[0,1]} \end{pmatrix} =$$

$$= -2^j \begin{pmatrix} c_k 2^{-j/2} 2\sqrt{2}\phi^{2,-}_s(2^jx-k)\chi_{[0,1]} \\ 2^{-j/2}\big((I-S)\phi^{1,-}_s(2^jx-k) - (I+S)\phi^{2,-}_s(2^jx-k)\big)\chi_{[0,1]} \end{pmatrix},$$

where $c_k = \sqrt{2}$ if $k = 0, 2^j$, and one otherwise. Remember that the functions had more symmetries. In Theorem 2 we proved that $\phi^{1,-}_s = \alpha + \beta$ and $\phi^{2,-}_s = \alpha - \beta$, where $\alpha = \phi^{2,-}_s\chi_{[-1,0]}$, $\beta = -\phi^{2,-}_s\chi_{[0,1]}$ and $\alpha(x) = \beta(-x)$. Notice that $2S\alpha = S\phi^{1,-}_s + S\phi^{2,-}_s$. With this in hand we see that

$$D\Phi_{s,j,k}(x) = 2^j \begin{pmatrix} -c_k 2\sqrt{2}\ 2^{-j/2}\phi^{2,-}_s(2^jx-k)\chi_{[0,1]} \\ 2^{-j/2}(2S\alpha(2^jx-k) - 2\beta(2^jx-k)) \end{pmatrix}.$$

For the interior scaling functions, that is when $i = 1, 2;\ \ k = 1, \ldots 2^j - 1$, we get:

$$\begin{pmatrix} D\phi^1_{s,j,k} \\ D\phi^2_{s,j,k} \end{pmatrix} = 2^j \begin{pmatrix} -2\sqrt{2}\phi^{2,-}_{s,j,k} \\ 2S\alpha_{j,k} - 2\beta_{j,k} \end{pmatrix} = -2^j T_M \Phi^{-}_{s,j,k};$$

and when $i = 2,\ \ k = 0$ we get

$$D\phi^2_{s,j,0} = 2^{j+1}(S\alpha_{j,0} - \beta_{j,0}) = 2^j(\phi^{1,-}_{s,j,1} + \phi^{2,-}_{s,j,1} + \sqrt{2}\phi^{2,-}_{s,j,0}).$$

And for the truly truncated ones, when $i=1$ and $k=0,2^j$, then:

$$D\phi^1_{s,j,k} = -2^{j+1}\sqrt{2}\phi^{2,-}_{s,j,k}.$$

□

Proof of Lemma 1 (b) (Commutation relations for the projections onto the approximating spaces.)

We want to show $D\tilde{P}^+_j = \tilde{P}_j D$ for all $j \geq 0$. We will deduce this identity from the explicit description given above of the derivatives of the scaling functions.

$$\begin{aligned}
D\tilde{P}^+_j f &= D\left(\sum_{i,k}\langle f, \phi^{i,-}_{s,j,k}\rangle \tilde{\phi}^{i,+}_{s,j,k}\right) \\
&= \sum_{k=1}^{2^j-1}\langle f, \Phi^-_{s,j,k}\rangle D\Phi^+_{\tilde{s},j,k} + \langle f, \phi^{2,-}_{s,j,0}\rangle D\phi^{2,+}_{\tilde{s},j,0} + \langle f, \phi^{2,-}_{s,j,2^j}\rangle D\phi^{2,+}_{\tilde{s},j,2^j} \\
&= \sum_{k=1}^{2^j-1}\langle f, \Phi^-_{s,j,k}\rangle 2^j T_{M^*}\Phi_{\tilde{s},j,k} + \langle f, \phi^{2,-}_{s,j,0}\rangle 2^j\sqrt{2}(2\phi^1_{\tilde{s},j,0} - \phi^2_{\tilde{s},j,0}) \\
&\quad + \langle f, \phi^{2,-}_{s,j,2^j}\rangle 2^j\sqrt{2}(2\phi^1_{\tilde{s},j,2^j} - \phi^2_{\tilde{s},j,2^j-1}).
\end{aligned}$$

Now we want to use the same summation by parts argument as before for the first summand, in the process we must add and subtract some contributions from the boundary terms and there will be some leftover which will be absorbed by the boundary terms, we obtain then:

$$\begin{aligned}
D\tilde{P}^+_j f &= \sum_{k=1}^{2^j-1}\langle f, -D\Phi_{s,j,k}\rangle\tilde{\Phi}_{s,j,k} - \langle f, 2^j\sqrt{2}\phi^{2,-}_{s,j,1}\rangle\phi^2_{\tilde{s},j,0} \\
&\quad - \langle f, 2^j\sqrt{2}\phi^{1,-}_{s,j,1}\rangle\phi^2_{\tilde{s},j,0} + \langle f, 2^j\sqrt{2}\phi^{2,-}_{s,j,2^j}\rangle\phi^2_{\tilde{s},j,2^j-1} \\
&\quad + \langle f, 2^j\sqrt{2}\phi^{2,-}_{s,j,0}\rangle(2\phi^1_{\tilde{s},j,0} - \phi^2_{\tilde{s},j,0}) \\
&\quad + \langle f, 2^j\sqrt{2}\phi^{2,-}_{s,j,2^j}\rangle(2\phi^1_{\tilde{s},j,2^j} - \phi^2_{\tilde{s},j,2^j-1}) \\
&= \sum_{k=1}^{2^j-1}\langle Df, \Phi_{s,j,k}\rangle\tilde{\Phi}_{s,j,k} + \langle Df, \phi^1_{s,j,0}\rangle\phi^1_{\tilde{s},j,0} \\
&\quad + \langle Df, \phi^1_{s,j,2^j}\rangle\phi^1_{\tilde{s},j,2^j} \\
&\quad + \langle Df, 2^j(-\sqrt{2}\phi^{2,-}_{s,j,0} - \phi^{1,-}_{s,j,1} - \phi^{2,-}_{\tilde{s},j,1})\rangle\phi^2_{\tilde{s},j,0}
\end{aligned}$$

the last summand is exactly $\langle f, -D\phi^2_{s,j,0}\rangle\phi^2_{\tilde{s},j,0} = \langle Df, \phi^2_{s,j,0}\rangle\phi^2_{\tilde{s},j,0}$. All together we get the claimed identity.

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J. Lakey
Dept. of Mathematics
New Mexico State University
Las Cruces, NM 88003-8001
jlakey@math.nmsu.edu

C. Pereyra
Dept. of Math. and Statistics
University of New Mexico
Albuquerque, NM 87106
crisp@math.unm.edu