Inferences on Correlation Coefficients of Bivariate Log-normal Distributions

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Abstract

This article considers inference on correlation coefficients of bivariate log-normal distributions. We developed generalized confidence intervals and hypothesis tests for the correlation coefficient, and extended the results for comparing two independent correlations. Simulation studies show that the suggested methods work well even for small samples. The methods are illustrated using two practical examples.

Key Words: bivariate log-normal, correlation coefficient, generalized confidence interval, generalized pivotal quantity, generalized p-value, hypothesis test.

1 Introduction

Log-normal distribution is a continuous probability distribution of a random variable whose logarithm is normally distributed. It is widely used to describe the distribution of positive random variables that exhibit skewness. The Pearson product-moment correlation is a well known measure of the strength and direction of linear relationship between two continuous random variables. This research concerns inference on correlation coefficients of bivariate log-normal distributions. Consider daily return of silver and gold funds following bivariate log-normal distribution during some period, we want to answer the question: what is the correlation between silver and gold? If in the following bear market, gold and silver started declines, are the correlation between silver and gold different from the past?

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Most research in literature concerns inference of a single log-normal mean, or comparing two independent log-normal means. Zhou, Li, Gao, and Tierney (2001) addressed the problem of comparing the means of a bivariate log-normal distribution. Krishnamoorthy and Mathew (2003) used generalized variables (GV) approach to compare two independent log-normal means. Chen and Zhou (2006) compared different methods for obtaining confidence intervals for the ratio (or difference) of two independent log-normal means, and concluded that the GV approach works better than other methods in providing the intended coverage. Using GV approach, Bebu and Mathew (2008) developed procedures of constructing a confidence interval for the ratio of bivariate log-normal means regardless of sample sizes. Recently, Lin (2014) compared the mean vectors of two independent multivariate log-normal distributions using GV approach.

The theory and application of Pearson correlation is well documented for normal and multivariate normal distributions. Inference on a correlation with a bivariate normal distribution can be tested by an exact t procedure or Fisher (1921)'s z transformation. When there are two samples, a common interest is to compare two independent or dependent correlations from the two samples. Olkin and Finn (1995) proposed a normal-based asymptotic result that can be used for testing two independent correlations. The problem of comparing two overlapping dependent correlations is relatively complicated. Hotelling (1940) first provided an exact conditional test. Williams (1959) proposed an unconditional method by modifying Hotelling's conditional test. Based on Neill and Dunn (1975)'s simulation studies, William's test was the best among 11 methods suggested in literature. Olkin and Finn (1990) derived an asymptotic result for hypothesis testing and confidence limits for the difference between two dependent correlations. Meng, Rosenthal, and Rubin (1992) proposed an asymptotic result for hypothesis testing based on Fisher's z transformations of the sample correlation coefficients. The test for comparing non-overlapping dependent correlations is first discussed by Pearson and Filon (1898). Tsui and Weerahandi (1989) introduced the concept of generalized p-value using GV approach for hypothesis testing. Later, Weerahandi (1993) discussed generalized confidence limits. Krishnamoorthy and Xia (2006) discussed inference on the correlation coefficients of a multivariate normal distribution using GV approach.

On the other hand, because of skewness, inferences on correlation of a bivariate log-normal distribution face with difficulties. There is little research found in literature. Lai, Cwrayner, and Hutchinson (1999) studied robustness of the sample correlation for the bivariate log-normal case. Their simulation studies indicated that the bias in estimating population correlation coefficient ρ of the bivariate log-normal distribution was very large if $\rho \neq 0$, and the bias could be reduced substantially only after three to four million of observations. Despite the difficulties, our research intends to fill the gap by providing a valid confidence interval and hypothesis tests for correlation coefficients of bivariate log-normal distributions. This paper is organized as follows. In Section 2, we review notations and generalized pivotal quantities for the elements of a variance-covariance matrix. In Section 3, we developed generalized confidence intervals (GCI) and hypothesis tests for correlation coefficient of a single sample and extend the results for comparing two independent correlations. In Section 4, we perform simulation studies. In Section 5, we give two examples to illustrate the use of the proposed methods. Finally, Section 6 gives the conclusion.

2 Notations and generalized pivotal quantities for the elements of a variance-covariance matrix

Let $\mathbf{Y}_1, \cdots, \mathbf{Y}_n$ be a random sample from a bivariate log-normal distribution, and let $\mathbf{X}_i = \ln \mathbf{Y}_i$ for $i = 1, 2, \cdots, n$. By definition, $\mathbf{X}_1, \mathbf{X}_2, \cdots, \mathbf{X}_n$ is a sample from a bivariate normal distribution with mean vector $\boldsymbol{\mu} = (\mu_1, \mu_2)'$, and variance-covariance matrix $\boldsymbol{\Sigma}$, i.e,.

$$\mathbf{X}_i \overset{iid}{\sim} N\left(\begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix}, \boldsymbol{\Sigma} = \begin{bmatrix} \sigma_{11}^2 & \sigma_{12}^2 \\ \sigma_{12}^2 & \sigma_{22}^2 \end{bmatrix} \right).$$

Let $\rho_X = \sigma_{21}^2 / \sqrt{\sigma_{11}^2 \sigma_{22}^2}$ be the population correlation of the bivariate normal distribution (note that ρ is used to denote the population correlation of the bivariate log-normal distribution), and leet $\mathbf{S}$ be the matrix of sums of squares of the cross-products,

$$\mathbf{S} = \sum_{i=1}^{n}(\mathbf{X}_i - \bar{\mathbf{X}})(\mathbf{X}_i - \bar{\mathbf{X}})' = \begin{bmatrix} S_{11} & S_{12} \\ S_{12} & S_{22} \end{bmatrix}. \tag{1}$$

Consider the problem of testing population correlation coefficient ρ of a bivariate log-normal distribution,

$$H_0 : \rho \leq \rho_0 \quad \text{vs.} \quad H_\alpha : \rho > \rho_0, \tag{2}$$

where ρ_0 is a specified value of ρ. We shall now give the definitions of generalized pivotal statistic $T_1(\mathbf{X};\mathbf{x},\rho;\eta)$ and generalized test statistic $T_2(\mathbf{X};\mathbf{x},\rho;\eta)$. Note that η denotes the nuisance parameter and may be more than one, and $\mathbf{x}$ is the observed value of $\mathbf{X}$.

Definition 1: To define a generalized confidence interval for parameter ρ, a generalized pivotal statistic $T_1(\mathbf{X};\mathbf{x},\rho,\eta)$ should satisfy the following two conditions:

(1) the distribution of the generalized pivotal statistic T_1 is free of any unknown parameters;

(2) the observed pivotal statistic $T_1(\mathbf{x};\mathbf{x},\rho,\eta)$ is the parameter of interest ρ.

The percentiles of $T_1(\mathbf{X};\mathbf{x},\rho,\eta)$ are used to construct a generalized confidence interval for ρ.

Definition 2: For the purpose of hypothesis testing, the generalized test variable for ρ is defined as $T_2 = T_1 - \rho$. The generalized test variable T_2 should satisfy the following three conditions:

(a) the distribution of T_2 is free of any unknown parameter;

(b) the observed value of T_2 is free of any unknown parameters;

(c) the distribution of T_2 is stochastically monotone in ρ.

If T_2 is stochastically increasing in ρ, The generalized p-value for testing the hypotheses in (2) is defined by $P = P[T_2(\mathbf{X};\mathbf{x},\rho,\eta) \geq T_2(\mathbf{x};\mathbf{x},\rho,\eta)|\rho = \rho_0]$. If T_2 is stochastically decreasing in ρ, The generalized p-value for testing the hypotheses in (2) is defined by $P = P[T_2(\mathbf{X};\mathbf{x},\rho,\eta) \leq T_2(\mathbf{x};\mathbf{x},\rho,\eta)|\rho = \rho_0]$.

As pointed out by Weerahandi (1993), the problem of finding an appropriate generalized pivotal quantity is a non-trival task. There is no systematic approach that can be used to find pivotal quantities for all problems. Interested readers may refer to Iyer and Patterson (2002) for generalized pivotal quantities of a large class of practical problems.

In the following, we will review generalized pivotal quantities for $\mathbf{\Sigma}$ (Bebu & Mathew, 2008). Note that the matrix of sums of squares of the cross-products $\mathbf{S}$ in (1) has a Wishart distribution with $\mathbf{\Sigma} = (\sigma_{ij})$ and degrees of freedom of $n-1$. Let $\sigma_{11}^* = \sigma_{11} - \sigma_{12}^2/\sigma_{22}$ and $S_{11}^* = S_{11} - S_{12}^2/S_{22}$. Using the fact that $\mathbf{S} \sim W_2(\mathbf{\Sigma}, n-1)$, the following three variables are independent and have either χ^2 or standard normal distribution (Johnson & Wichern, 2008): $V_{22} = S_{22}/\sigma_{22} \sim \chi^2_{n-1}$, $V_{11}^* = S_{11}^*/\sigma_{11}^* \sim \chi^2_{n-2}$, and $Z = (S_{12} - \dfrac{\sigma_{12}}{\sigma_{22}} S_{22})/\sqrt{\sigma_{11}^* S_{22}} \sim N(0,1)$. Let $\mathbf{s} = (s_{ij})$ be the observed $\mathbf{S}$. We define

$$b_{22} = \frac{\sigma_{22}}{S_{22}} s_{22} = \frac{s_{22}}{V_{22}}, \tag{3}$$

$$\begin{aligned} b_{12} &= \frac{\sigma_{22}}{S_{22}} s_{12} - [\sqrt{s_{11}^* s_{22}} \frac{S_{12} - \frac{\sigma_{12}}{\sigma_{22}} S_{22}}{\sqrt{\sigma_{11}^* S_{22}}} \sqrt{\frac{\sigma_{11}^*}{S_{11}^*}} \frac{\sigma_{22}}{S_{22}} \\ &= \frac{s_{12}}{V_{22}} - \sqrt{s_{11}^* s_{22}} \frac{Z}{\sqrt{V_{11}^*}} \frac{1}{V_{22}}, \end{aligned} \tag{4}$$

and

$$\begin{aligned} b_{11} &= \frac{\sigma_{11}^*}{S_{11}^*} s_{11}^* + \frac{b_{12}^2}{b_{22}} \\ &= \frac{s_{11}^*}{V_{11}^*} + \frac{b_{12}^2}{b_{22}}. \end{aligned} \tag{5}$$

It is easy to show that b_{ij}s are free of any parameters, and the observed value of b_{ij}s are σ_{ij}s. Therefore, $\mathbf{B} = (b_{ij})$ are the generalized pivotal quantities of the covariance matrix $\mathbf{\Sigma} = (\sigma_{ij})$. If $h(\mathbf{\Sigma})$ is a real-valued function of $\mathbf{\Sigma}$, it is easy to show that $h(\mathbf{B})$ is a generalized pivotal variable and $h(\mathbf{B}) - \rho$ is a generalized test variable for ρ if the distribution of $h(\mathbf{\Sigma}) - \rho$ is stochastically monotone in ρ.

3 Inference on a single correlation coefficient

In this section, we consider hypothesis tests and interval estimation for the population correlation coefficient ρ from a bivariate log-normal distribution. We shall concern ourselves first on inference of a single correlation, then on comparison of two independent correlations.

3.1 Inference on a single correlation coefficient

Let $\mathbf{Y}_i = (Y_{i1}, Y_{i2})' \overset{iid}{\sim}$ bivariate log-normal distribution, i.e,

$$\begin{bmatrix} X_{i1} \\ X_{i2} \end{bmatrix} = \begin{bmatrix} \ln Y_{i1} \\ \ln Y_{i2} \end{bmatrix} \overset{iid}{\sim} N\left(\begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix}, \mathbf{\Sigma} = \begin{bmatrix} \sigma_{11}^2 & \sigma_{12}^2 \\ \sigma_{12}^2 & \sigma_{22}^2 \end{bmatrix} \right).$$

Let ρ be the correlation coefficient between Y_{i1} and Y_{i2}, i.e,

$$\rho = \frac{\text{cov}(Y_{i1}, Y_{i2})}{\sqrt{\text{Var}(Y_{i1})\text{Var}(Y_{i2})}}.$$

Using the facts that $E(Y_{i1}) = e^{\mu_1+\sigma_{11}^2/2}, E(Y_{i2}) = e^{\mu_2+\sigma_{22}^2/2}$, $V(Y_{i1}) = e^{(2\mu_1+\sigma_{11}^2)}(e^{\sigma_{11}^2} - 1)$ and $V(Y_{i2}) = e^{(2\mu_2+\sigma_{22}^2)}(e^{\sigma_{22}^2} - 1)$, we can show that

$$\text{Cov}(Y_{i1}, Y_{i2}) = e^{\mu_1+\mu_2+(\sigma_{11}^2+\sigma_{22}^2)/2}(e^{\sigma_{12}^2} - 1),$$

and

$$\sqrt{\text{Var}(Y_{i1})\text{Var}(Y_{i2})} = e^{(\mu_1+\mu_2)+(\sigma_{11}^2+\sigma_{22}^2)/2}\sqrt{(e^{\sigma_{11}^2} - 1)(e^{\sigma_{22}^2} - 1)}.$$

Therefore,

$$\rho = \frac{e^{\sigma_{12}^2} - 1}{\sqrt{(e^{\sigma_{11}^2} - 1)(e^{\sigma_{22}^2} - 1)}}. \tag{6}$$

The generalized pivotal variable G_ρ is given by

$$G_\rho = h(\mathbf{B}) = (e^{b_{12}} - 1)/\sqrt{(e^{b_{11}} - 1)(e^{b_{22}} - 1)}. \tag{7}$$

The generalized test variable for ρ is $G_\rho^t = G_\rho - \rho$, which is stochastically decreasing in ρ. It is easy to show that the generalized p-value for testing the hypotheses in (2) is the same as $P(G_\rho \leq \rho_0)$. Reject H_0 in (2) when the generalized p-value is less than α. The following algorithm is developed to estimate the generalized confidence limits and the generalized p-values.

Algorithm 1:

1. For a given value of $(\mathbf{y}_1, \cdots, \mathbf{y}_n)$ ($\mathbf{y}_i$ is the observed value of $\mathbf{Y}_i$), compute $(\mathbf{x}_1, \cdots, \mathbf{x}_n) = (\ln\mathbf{y}_1, \cdots, \ln\mathbf{y}_n)$ and

$$\mathbf{s} = \sum_{i=1}^{n}(\mathbf{x}_i - \bar{\mathbf{x}})(\mathbf{x}_i - \bar{\mathbf{x}})' = \begin{bmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{bmatrix};$$

2. For $l = 1, 2, \cdots, L$, generate $V_{22} \sim \chi^2_{n-1}, V^*_{11} \sim \chi^2_{n-2}$, $Z \sim N(0,1)$, compute b_{22}, b_{12}, b_{11} and G_ρ by Equations (3), (4), (5) and (7) respectively;

3. Let $Q_l = 1$ if $G_\rho \leq \rho_0$;

(end loop)

$\sum_{l=1}^{L} Q_l/L$ is a Monte carlo estimate of the generalized p-value for testing (2). Similarly we can derive the generalized p-value for right-sided test. The generalized two-sided confidence interval for the correlation coefficient ρ can be constructed by using $100(\alpha/2)$th and $100(1-\alpha/2)$th percentiles of G_ρ as the confidence limits. Similarly, the $100(1-\alpha)$th percentile of the G_ρ is a $100(1-\alpha)$th lower limit for ρ. The 100αth percentile of the G_ρ is a $100(\alpha)$th upper limit for ρ.

3.2 Comparison between two independent correlation coefficients

The problem of comparing correlations of different groups also attracts a lot of interest. For example, it may be of interest to see if the correlation between silver and gold is lower in a bull market than that of a bear market. In the following, we use superscript (k) to denote group $k, k = 1, 2$. A general hypotheses test between correlations $\rho^{(1)}$ and $\rho^{(2)}$ from the two groups can be described as

$$H_0 : \rho^{(1)} - \rho^{(2)} \leq c \quad \text{vs.} \quad H_\alpha : \rho^{(1)} - \rho^{(2)} > c, \tag{8}$$

where c is a constant. In this section, we extend the results in Section 3.1 to two independent bivariate log-normal distributions.

Let $\mathbf{Y}_i^{(k)} = (Y_{i1}^{(k)}, Y_{i2}^{(k)})' \overset{iid}{\sim}$ bivariate log-normal distribution for $k = 1, 2$ and $i = 1, 2, \cdots, n_k$,

$$\begin{bmatrix} X_{i1}^{(k)} \\ X_{i2}^{(k)} \end{bmatrix} = \begin{bmatrix} \log Y_{i1}^{(k)} \\ \log Y_{i2}^{(k)} \end{bmatrix} \sim N\left(\begin{bmatrix} \mu_1^{(k)} \\ \mu_2^{(k)} \end{bmatrix}, \mathbf{\Sigma}^{(k)} = \begin{bmatrix} \sigma_{11}^{(k)} & \sigma_{12}^{(k)} \\ \sigma_{21}^{(k)} & \sigma_{22}^{(k)} \end{bmatrix} \right).$$

The matrix of sums of squares and cross-products $\mathbf{S}^{(k)}$ is

$$\mathbf{S}^{(k)} = \sum_{j=1}^{n_k} (\mathbf{X}_j^{(k)} - \bar{\mathbf{X}}^{(k)})(\mathbf{X}_j^{(k)} - \bar{\mathbf{X}}^{(k)})', k = 1, 2.$$

Let $G_\rho^{(k)}$ be a generalized pivotal variable for $\rho^{(k)}, k = 1, 2$. The generalized pivotal variable for $\rho^{(1)} - \rho^{(2)}$ can be obtained using the results in Equation (7) as the following

$$G_{\rho_{12}} = G_\rho^{(1)} - G_\rho^{(2)} = \frac{e^{b_{12}^{(1)}} - 1}{\sqrt{(e^{b_{11}^{(1)}} - 1)(e^{b_{22}^{(1)}} - 1)}} - \frac{e^{b_{12}^{(2)}} - 1}{\sqrt{(e^{b_{11}^{(2)}} - 1)(e^{b_{22}^{(2)}} - 1)}},$$

where $b_{ij}^{(1)}$ is the pivotal quantities calculated from group 1, and $b_{ij}^{(2)}$ is the pivotal quantities calculated from group 2. The generalized test variable for $\rho^{(1)} - \rho^{(2)}$ is $G_{\rho_{12}}^t = G_\rho^{(1)} - G_\rho^{(2)} - (\rho^{(1)} - \rho^{(2)})$, which is stochastically decreasing in $\rho^{(1)} - \rho^{(2)}$. The generalized p-value for testing the hypotheses in (8) is the same as $P(G_{\rho_{12}} \leq c)$. Reject H_0 in (8) when the generalized p-value is less than α. The following algorithm is developed to estimate the percentiles of $G_{\rho_{12}}$ and generalized p-values:

Algorithm 2:

1. For a given $(\mathbf{y}_1^{(1)}, \cdots, \mathbf{y}_n^{(1)})$ and $(\mathbf{y}_1^{(2)}, \cdots, \mathbf{y}_m^{(2)})$, compute $(\mathbf{x}_1^{(1)}, \cdots, \mathbf{x}_n^{(1)}) = (\ln\mathbf{y}_1^{(1)}, \cdots, \ln\mathbf{y}_n^{(1)})$, $(\mathbf{x}_1^{(2)}, \cdots, \mathbf{x}_m^{(2)}) = (\ln\mathbf{y}_1^{(2)}, \cdots, \ln\mathbf{y}_m^{(2)})$ and $\mathbf{s}^{(k)} = \sum(\mathbf{x}_i^{(k)} - \bar{\mathbf{x}}^{(k)})(\mathbf{x}_i^{(k)} - \bar{\mathbf{x}}^{(k)})', k = 1, 2;$

2. For $l = 1, 2, \cdots, L$, generate $V_{22}^{(1)} \sim \chi^2_{n-1}$, $V_{22}^{(2)} \sim \chi^2_{m-1}$, $V_{11}^{*(1)} \sim \chi^2_{n-2}$, $V_{11}^{*(2)} \sim \chi^2_{m-2}$, $Z^{(1)} \sim N(0, 1), Z^{(2)} \sim N(0, 1);$

3. For $k = 1, 2$, compute

$$b_{22}^{(k)} = \frac{s_{22}^{(k)}}{V_{22}^{(k)}}, \quad b_{12}^{(k)} = \frac{s_{12}^{(k)}}{V_{22}^{(k)}} - \left[\sqrt{s_{11}^{*(k)} s_{22}^{(k)}} \frac{Z^{(k)}}{\sqrt{V_{11}^{*(k)}}} \frac{1}{V_{22}^{(k)}}\right]$$

$$b_{11}^{(k)} = \frac{s_{11}^{*(k)}}{V_{11}^{*(k)}} + \frac{{b_{12}^{(k)}}^2}{b_{22}^{(k)}}$$

$$G_\rho^{(k)} = \frac{e^{b_{12}^{(k)}} - 1}{\sqrt{(e^{b_{11}^{(k)}} - 1)(e^{b_{22}^{(k)}} - 1)}} \quad \text{and} \quad G_{\rho_{12}} = G_\rho^{(1)} - G_\rho^{(2)};$$

4. Let $C_l = 1$ if $G_{\rho_{12}} \leq c$;

(end loop)

$\sum_{l=1}^{L} C_l/L$ is a Monte carlo estimate of the generalized p-value for testing (8). Similarly we can derive the generalized p-value for right-sided test. The generalized two-sided confidence interval for $\rho^{(1)}-\rho^{(2)}$ can be constructed by using $100(\alpha/2)$th and $100(1-\alpha/2)$th percentiles of $G_{\rho_{12}}$ as the confidence limits. The generalized left-sided and right-sided confidence intervals for $\rho^{(1)} - \rho^{(2)}$ can be constructed by using $100(1-\alpha)$th and $100(\alpha)$th percentiles of $G_{\rho_{12}}$ respectively as the confidence limits.

4 Simulation studies

In this section, a small simulation study was conducted to evaluate the proposed GCI and hypothesis tests. The simulation set up follows from Bebu and Mathew (2008) with factors: (1) mean vector $\mu_1 = \mu_2 = 0$; (2) sample sizes: $n = 5, n = 10$ and $n = 20$; (3) normal correlation coefficients: $\rho_X = -0.9, 0.1$ and 0.9; (4) variance-covariance diagonal elements: $(\sigma_{11}, \sigma_{22}) = (1,5), (5,5)$ and $(1,10)$ (note that ρ_X, σ_{11} and σ_{22} determine the covariance matrix $\mathbf{\Sigma}$, the specified correlation coefficient ρ_0 of the bivariate log-normal distribution is calculated using Equation (6)); (5) tests considered: two-sided, left-sided and right-sided; and (6) nominal levels: 0.01, 0.05 and 0.1.

Simulation does $L = 10000$ times for each setting. Algorithms 1 and 2 are used to calculate the GCI and generalized p-values of simple correlation coefficients and comparison between two correlation coefficients respectively. Tables 1 and 2 report the simulated coverage levels regarding a simple correlation coefficient. Tables 3 and 4 report the simulated coverage levels from comparison between two independent correlation coefficients. For two-sided test, we also reported left errors and right errors. Interestingly, left and right errors are roughly the same when comparing two independent correlations. However, the shape of error is related to the sign of correlation coefficient when testing for a simple correlation coefficient. We observe that right error is much larger than left error, when correlation coefficient is positive. For example, under the setting of $n = 5, \rho_X = 0.9$, and $(\sigma_{11}, \sigma_{22}) = (5, 5)$, left error is only 0.0004, while right error is 0.0313. On the other hand, if the correlation coefficient is negative, left error is much larger than the right error. The simulation results show that coverage is acceptable when sample sizes are 5 or 10, and the coverage almost reaches the nominal level when sample size is 20. The proposed methods work well for correlation coefficients of the bivariate log-normal distributions.

5 Examples

5.1 Example on quantitative assay problem

Hawkins (2002) investigated 56 assay pairs for cyclosporin from blood samples of organ transplant recipients obtained by a standard approved method: high-performance liquid chromatography (HPLC) and an alternative radio-immunoassay (RIA) method. Hawkins (2002) showed that data followed a bivariate log-normal distribution. Using our proposed method, we want to test if the correlation between the two methods are linearly correlated. The estimated variance-covariance matrix is found to be

$$\mathbf{s} = \begin{pmatrix} 22.5608 & 19.3732 \\ 19.3732 & 18.9951 \end{pmatrix}.$$

Using algorithm 1, a two sided 95% generalized confidence interval is (0.8732,0.9501), which doesn't include 0. We conclude that cyclosporin from the two methods are highly correlated.

5.2 Example on financial data

A popular financial model is the well known geometric Brownian motion process,

$$P(t) = P_0 * e^{Y(t)},$$

where $P(t)$ is the price of a stock at time t, P_0 is the initial price of the stock (or fund) and $Y(t) > 0$ is a Brownian motion process with drift coefficient $\mu > 0$ and variance parameter σ^2. The interest of study is the correlation of daily return $P(t)/P(t-1)$ of silver and daily return of gold. We investigated two exchange traded funds, Shares Silver Trust (SLV) and SPDR Gold Shares (GLD), whose net assets are 6.6 billion and 33.9613 billion respectively.

Table 1: The simulated coverage levels of two-sided GCI for a simple correlation coefficient. "C" denotes the simulated coverage probabilities, "LE" denotes the simulated left error and "RE" is the simulated right error. "LE + RE = 1-C".

				$\alpha = 0.01$			$\alpha = 0.05$			$\alpha = 0.1$		
n	ρ_X	σ_{11}	σ_{22}	C	LE	RE	C	LE	RE	C	LE	RE
5	-0.9	1	5	.9928	.0072	.0000	.967	.0326	.0004	.9371	.0597	.0032
		5	5	.9935	.0065	0	.9686	.0311	.0003	.9308	.0675	.0017
		1	10	.9930	.00700	0	.9700	.0288	.0012	.9402	.0551	.0047
	0.1	1	5	.9849	.0005	.0146	.9397	.0039	.0564	.8845	.0128	.1027
		5	5	.9735	.0004	.0261	.9412	.0013	.0575	.8970	.0010	.1020
		1	10	.9838	.0001	.0161	.9290	.0039	.0671	.8694	.0088	.1218
	0.9	1	5	.9927	.0003	.0070	.9556	.0036	.0408	.9167	.0061	.0772
		5	5	.9682	.0004	.0313	.9420	.0019	.0561	.8970	.0061	.0969
		1	10	.9945	.0001	.0054	.9686	.0032	.0282	.9278	.0111	.0611
10	-0.9	1	5	.9938	.0057	.0005	.9635	.0289	.0076	.9197	.0547	.0256
		5	5	.9936	.0063	.0001	.9628	.0312	.0060	.9209	.0579	.0212
		1	10	.9938	.0058	.0004	.9597	.0272	.0131	.9181	.0513	.0306
	0.1	1	5	.9886	.0017	.0097	.9446	.0100	.0454	.8998	.0212	.0790
		5	5	.9823	.0008	.0169	.9270	.0087	.0643	.8717	.0173	.1110
		1	10	.9870	.0003	.0127	.9358	.0056	.0586	.8859	.0120	.1021
	0.9	1	5	.9935	.0006	.0059	.9640	.0058	.0302	.9192	.0171	.0637
		5	5	.9817	.0015	.0168	.9234	.0054	.0712	.9010	.0123	.0867
		1	10	.9921	.0013	.0066	.9589	.0143	.0268	.9093	.0310	.0597
20	-0.9	1	5	.9911	.0050	.0039	.9532	.0263	.0205	.9054	.0528	.0418
		5	5	.9926	.0046	.0028	.9511	.0278	.0211	.9036	.0533	.0431
		1	10	.9899	.0052	.0049	.9501	.0253	.0246	.9005	.0498	.0497
	0.1	1	5	.9900	.0028	.0072	.9450	.0148	.0402	.9031	.0289	.068
		5	5	.9870	.0023	.0107	.9392	.011	.0498	.8878	.0256	.0866
		1	10	.9871	.0009	.0120	.9479	.0077	.0444	.8861	.0178	.0961
	0.9	1	5	.9911	.0021	.0068	.9544	.0149	.0307	.9114	.0306	.0580
		5	5	.9830	.0025	.0145	.9382	.0085	.0533	.8849	.0235	.0916
		1	10	.9894	.0038	.0068	.9555	.0202	.0243	.9038	.0454	.0508

Table 2: The simulated coverage levels of one-sided tests for a simple correlation coefficient.

				$\alpha = 0.01$		$\alpha = 0.05$		$\alpha = 0.1$	
n	ρ_X	σ_{11}	σ_{22}	left-sided	right-sided	left-sided	right-sided	left-sided	right-sided
5	-0.9	1	5	1.0000	.9885	.9959	.9385	.9847	.8753
		5	5	1.0000	.9892	.9989	.9295	.9887	.8649
		1	10	1.0000	.9879	.9968	.9452	.9744	.8837
	0.1	1	5	.9739	.9981	.8913	.9868	.8091	.9693
		5	5	.9558	.9989	.9044	.9883	.8500	.9748
		1	10	.9702	.9994	.9006	.9919	.8830	.9744
	0.9	1	5	.9868	.9993	.9283	.9927	.8504	.9780
		5	5	.9486	.9991	.9004	.9902	.8567	.9775
		1	10	.9891	.9994	.9432	.9893	.8855	.9652
10	-0.9	1	5	.9991	.9875	.9744	.9456	.9287	.8897
		5	5	.9991	.9861	.9762	.9396	.9343	.8795
		1	10	.9976	.9882	.9683	.9475	.9146	.8960
	0.1	1	5	.9809	.9971	.9164	.9807	.8483	.9500
		5	5	.9717	.9981	.8775	.9833	.8632	.9561
		1	10	.9775	.9981	.8973	.9878	.8107	.9665
	0.9	1	5	.9887	.9982	.9384	.9833	.8767	.9576
		5	5	.9689	.9982	0.9398	.9822	0.8890	.9615
		1	10	.9900	.9967	.9453	.9679	.8919	.9229
20	-0.9	1	5	.9923	.9905	.9560	.9469	.9032	.8906
		5	5	.9921	.9876	.9560	.9421	.9107	.8887
		1	10	.9916	.9892	.9529	.9468	.8994	.8958
	0.1	1	5	.9838	.9960	.9418	.9624	.8699	.9123
		5	5	.9803	.9948	.9481	.9594	.8980	.9152
		1	10	.9788	.9983	.9468	.9776	.8959	.9114
	0.9	1	5	.9887	.9961	.9403	.9709	.8822	.9103
		5	5	.9795	.9968	.9464	.9700	.8900	.9057
		1	10	.9903	.9904	.9448	.9552	.8983	.8997

Table 3: The simulated coverage levels of two-sided GCI for a comparison between two independent correlation coefficients. "C" denotes the simulated coverage probabilities, "LE" denotes the simulated left error and "RE" is the simulated right error. "LE + RE = 1-C".

				$\alpha = 0.01$			$\alpha = 0.05$			$\alpha = 0.1$		
n	$\rho_{X1} = \rho_{X2}$	σ_{11}	σ_{22}	C	LE	RE	C	LE	RE	C	LE	RE
5	-0.9	1	5	.9999	.0001	0	.9974	.0016	.0010	.9850	.0067	.0083
		5	5	1	0	0	.9970	.0017	.0013	.9881	.0056	.0063
		1	10	.9998	.0002	0	.9963	.0013	.0024	.9805	.0093	.0102
	0.1	1	5	.9968	.0016	.0016	.9748	.0129	.0123	.9376	.0304	.0320
		5	5	.9963	.0019	.0018	.9703	.0152	.0145	.9384	.0322	.0294
		1	10	.9978	.0010	.0012	.9813	.0108	.0079	.9487	.0254	.0259
	0.9	1	5	.9986	.0007	.0007	.9862	.0071	.0067	.9655	.0178	.0167
		5	5	.9966	.0015	.0019	.9775	.0117	.0108	.9443	.0292	.0265
		1	10	.9992	.0004	.0004	.9885	.0060	.0055	.9681	.0140	.0179
10	-0.9	1	5	.9980	.0008	.0012	.9761	.0132	.107	.9311	.0341	.0348
		5	5	.9983	.0005	.0012	.9749	.0128	.0123	.9318	.0340	.0342
		1	10	.9979	.0010	.0011	.9684	.0155	.0161	.9210	.0388	.0402
	0.1	1	5	.9963	.0020	.0017	.9697	.0145	.0158	.9281	.0369	.0350
		5	5	.9958	.0015	.0027	.9706	.0141	.0153	.9359	.0320	.0321
		1	10	.9983	.0006	.0011	.9772	.0115	.0113	.9479	.0257	.0264
	0.9	1	5	.9965	.0015	.0020	.9756	.0124	.0120	.9436	.0298	.0266
		5	5	.9949	.0028	.0023	.9666	.0166	.0168	.9255	.0377	.0368
		1	10	.9962	.002	.0018	.9650	.0175	.0175	.9247	.0358	.0395
20	-0.9	1	5	.9913	.0041	.0046	.9515	.0235	.0250	.9043	.0467	.0490
		5	5	.9907	.0048	.0045	.9511	.0251	.0238	.9081	.0462	.0457
		1	10	.9907	.0053	.0040	.9527	.0226	.0247	.8984	.0499	.0517
	0.1	1	5	.9941	.0028	.0031	.9630	.0196	.0174	.9151	.0421	.0428
		5	5	.9943	.0026	.0031	.9646	.0166	.0188	.9185	.0431	.0384
		1	10	.9976	.0012	.0012	.9754	.0124	.0122	.9397	.0304	.0299
	0.9	1	5	.9938	.0029	.0033	.9637	.0171	.0192	.9265	.0356	.0379
		5	5	.9918	.0037	.0045	.9566	.0214	.0220	.9143	.0438	.0419
		1	10	.9903	.0051	.0046	.9554	.0202	.0244	.9033	.0476	.0491

Table 4: The simulated coverage levels of one-sided tests for a comparison between two independent correlation coefficients.

				$\alpha = 0.01$		$\alpha = 0.05$		$\alpha = 0.1$	
n	$\rho_{X1} = \rho_{X2}$	σ_{11}	σ_{22}	left-sided	right-sided	left-sided	right-sided	left-sided	right-sided
5	-0.9	1	5	.9996	.9999	.9951	.9928	.9670	.9659
		5	5	1.0000	1.0000	.9943	.9938	.9722	.9717
		1	10	.9999	.9998	.9893	.9923	.9617	.9601
	0.1	1	5	.9956	.9968	.9715	.9691	.9319	.9269
		5	5	.9952	.9963	.9652	.9669	.9280	.9253
		1	10	.9973	.9976	.9772	.9779	.9413	.9365
	0.9	1	5	.9982	.9986	.9848	.9835	.9535	.9547
		5	5	.9967	.9970	.9690	.9677	.9266	.9328
		1	10	.9993	.9990	.9843	.9820	.9503	.9488
10	-0.9	1	5	.9970	.9974	.9630	.9616	.9177	.9154
		5	5	.9978	.9978	.9647	.9672	.9176	.9172
		1	10	.9963	.9969	.9614	.9617	.9092	.9101
	0.1	1	5	.9946	.9944	.9666	.9609	.9205	.9187
		5	5	.9943	.9940	.9648	.9635	.9187	.9199
		1	10	.9976	.9977	.9740	.9737	.9327	.9318
	0.9	1	5	.9967	.9965	.9715	.9720	.9269	.9297
		5	5	.9927	.9936	.9614	.9625	.9151	.9121
		1	10	.9949	.9923	.9618	.9636	.9157	.9123
20	-0.9	1	5	.9897	.9918	.9495	.9526	.9053	.9032
		5	5	.9912	.9913	.9518	.9530	.9039	.9023
		1	10	.9902	.9884	.9524	.9507	.9012	.9038
	0.1	1	5	.9936	.9927	.9599	.9581	.9137	.9152
		5	5	.9935	.9932	.9589	.9641	.9125	.9132
		1	10	.9973	.9970	.9716	.9690	.9309	.9331
	0.9	1	5	.9937	.9938	.9578	.9631	.9138	.9115
		5	5	.9928	.9924	.9580	.9555	.9100	.9097
		1	10	.9907	.9907	.9522	.9505	.9045	.9039

The first period we studied is from August 27th 2010 to April 18th 2011 with $n = 161$ trading days, when the bull market was observed. The sequence of daily return of each fund consists 160 records. It is well understood that the two sequences of daily returns follow bivariate log-normal distributions. The estimated variance-covariance matrix is found to be

$$\mathbf{s}^{(1)} = \begin{pmatrix} 0.06272876 & 0.02584196 \\ 0.02584196 & 0.01405631 \end{pmatrix}.$$

Using algorithm 1, a two sided 95% confidence interval is computed as $(0.8271, 0.9027)$. The correlation between silver and gold when the bubble of commodities precious metals happened is significantly different from zero. As we can see from the confidence interval, the correlation is quite high.

After crash in May 2011, gold and silver started declines. The second period we studied is from February 18th 2013 to December 15th 2013 when a bear market was observed. We are wondering if the correlation between silver and gold during the bull market period will be different from the bear market period. The estimated variance-covariance during this period is found to be

$$\mathbf{s}^{(2)} = \begin{pmatrix} 0.10307072 & 0.06233194 \\ 0.06233194 & 0.04480097 \end{pmatrix}.$$

Using algorithm 1, a 95% CI is found to be $(0.8918, 0.9354)$. The correlation between silver and gold after the market crash is still pretty high. Using algorithm 2, a comparison of correlations between these two periods $\rho^{(2)} - \rho^{(1)}$ gives a two sided confidence interval as $(0.0061, 0.0941)$. These finding are interesting. In both bull and bear market, silver and gold have high correlations, and the correlation is stronger in a bear market than in a bull market.

6 Conclusions

The skewness of the log-normal distribution brings difficulty on inference of correlation coefficients of bivariate log-normal distributions, particularly for small samples. Our research fills this gap by providing GCIs and hypothesis tests using a GV approach. We also developed tests for comparing two independent correlations. The properties of the suggested methods are evaluated by simulation studies and have been shown to be satisfactory even for small samples. Example on quantitative assay problem shows that correlation between cyclosporin from a standard approved method high-performance liquid chromatography (HPLC) and an alternative radio-immunoassay (RIA) method are pretty high. Another example on financial daily return data shows that for silver and gold, the correlation is quite high, and the correlation is stronger in a bear market than in a bull market.

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