Math 1512 – Final Exam Review – Fall 2025

Topics and sample problems

A. Definitions

Continuity.

1. State the definition of continuity.
2. Let

$$F(r) = \begin{cases} GMr/R^3 & \text{if} \quad 0 \le r < R \\ GM/r^2 & \text{if} \quad r \ge R \end{cases}$$

where G, M are positive constants. Use the definition to show that F is continuous at $r = R$.

The derivative.

3. State the definition of the derivative. Sketch a graph that illustrates
4. Use the definition to find the derivative of

(a) $f(x) = \sqrt{x+2}$ (b) $f(x) = 1/x^2$ (c) $f(x) = x^3$

5. True or False: If f is continuous at $x = a$, then it is differentiable at a. If false, give counterexample.
6. Recognize when a limit is a derivative.

The definite integral.

7. State the definition as a limit of a Riemann sum.
8. *Rewriting the limit of a sum as an integral.*

Evaluate $\lim_{n\to\infty} \sum_{i=1}^{n} \sqrt{x_i}\Delta x$, where $\{x_i\}$ is a partition of $[0,1]$. (Hint: write as an integral.)

B. Using the Rules

Limits. Find limits as $x \to c$, one-sided limits, infinite limits (vertical asymptotes), limits at infinity. Including functions whose graphs have holes, functions defined piecewise. Find limits using a graph.

9. Find the following limits. Show all work. Use the symbols ∞ or $-\infty$ if appropriate.

(a) $\lim_{x\to\infty} \frac{x^3+3x+1}{3x^3+5}$

(b) $\lim_{x\to-5} \frac{x^2-25}{x+5}$

(c) $\lim_{x\to5^-} \frac{x+1}{x-5}$

(d) $\lim_{x\to1} \frac{x^3-3x^2+3x-1}{x-1}$

(e) $\lim_{x\to-2^+} \frac{x-1}{x^2(x+2)}$

(f) $\lim_{x\to9} \frac{9-x}{3-\sqrt{x}}$

10. §1.5: 4-9 (find limits and function values using the given graphs)

Derivatives. Use power rule, product rule, quotient rule, chain rule, to find derivatives. Know derivatives of x^p, $\sin(x)$, $\cos(x)$, $\tan(x)$, $\sec(x)$, $\cot(x)$, $\csc(x)$. Know implicit differentiation. Be able to apply the Fundamental Theorem of Calculus Part 1.

11. Chapter 2 Review Exercises: 13-44 odd
12. Find the derivatives of the following functions.

(a) $f(x) = x\sin(\pi x)$

(b) $f(t) = \cos(\tan t)$

(c) $g(s) = \sqrt{s} + \dfrac{1}{\sqrt[3]{s^4}}$

(d) $f(x) = \displaystyle\int_{x^2}^{2} \sqrt{t^2+1}\,dt$

(e) $f(x) = \sec(x^3+1)$

(f) $s(t) = \dfrac{t}{1-t^2}$

(g) $g(s) = \displaystyle\int_{2s}^{3s} \frac{u}{u^2+1}\,du$

(h) $P(R) = \dfrac{E^2R}{(R+r)^2}$, where E, r are constants

13. Find dy/dx.

(a) $y = 2x\sqrt{x^2+1}$

(b) $y = \dfrac{3x-2}{\sqrt{2x+1}}$

(c) $y = \tan^2 x$

(d) $xy^4 + x^2y = x + 3y$

(e) $\sin(xy) = x^2 - y^2$

14. If f and g are differentiable, find

(a) $\dfrac{d}{dx}\left[\sqrt{f(x)}\right]$

(b) $\dfrac{d}{dx}\left[f(\sqrt{x})\right]$

(c) $\dfrac{d}{dx}\left[\sqrt{x}f(x)\right]$

(d) $\dfrac{d}{dx}\left[f(g(x))\right]$

(e) $\dfrac{d}{dx}\left[f(f(x))\right]$

(f) $\dfrac{d}{dx}\left[\sqrt{\dfrac{f(x)}{g(x)}}\right]$

15. Let

$$F(r) = \begin{cases} GMr/R^3 & \text{if } \; 0 \le r < R \\ GM/r^2 & \text{if } \; r \ge R \end{cases}$$

where G, M, and R are positive constants. Is F differentiable at $r = R$?

Indefinite integrals. Find antiderivatives (when possible) either directly or using substitution. Always check your answer!

16. §3.9: 53, 55 (graphing antiderivative, including of piecewise)
17. §4.5: 9-32 odd

Definite integrals. (Fundamental Theorem of Calculus - Part 2). Find definite integrals. When using substitution, change the bounds of integration.

18. §4.5: 37-52 odd
19. Chapter 4 Review Exercises: 11-32 odd

The function $g(x) = \int_a^x f(t)\,dt$**.** What does $g(x)$ represent? What is $g'(x)$? (use FTC - Part 1)

20. (a) What is the difference between $\int f(x)\,dx$, $\int_a^x f(t)\,dt$, and $\int_a^b f(t)\,dt$?

    (b) What is the difference between $\frac{d}{dx}\left[\int_a^x f(t)\,dt\right]$ and $\frac{d}{dx}\left[\int_b^x f(t)\,dt\right]$, if any?
21. Find a function f and a value of the constant a such that $2\int_a^x f(t)\,dt = 2\sin x - 1$.
22. Given the equation $\quad x\sin(\pi x) = \displaystyle\int_0^{x^2} f(t)\,dt$, find $f(4)$.
23. Evaluate

(a) $\displaystyle\int_0^1 \frac{d}{dx}\left[\frac{1}{1+x^2}\right]dx$

(b) $\displaystyle\frac{d}{dx}\left[\int_0^1 \frac{1}{1+x^2}\,dx\right]$

(c) $\displaystyle\frac{d}{dx}\left[\int_0^x \frac{1}{1+t^2}\,dt\right]$

(d) $\displaystyle\frac{d}{dx}\left[\int_{\sqrt{x}}^1 \frac{1}{1+t^2}\,dt\right]$

24. Chapter 4 Review Exercises: 9, 10

C. Applications of the derivative

Geometric interpretation of the derivative,

25. Chapter 2 Review Exercises: 49-50 (just tangent line), 53, 54 (implicit differentiation)

Graphing.

26. Sketch the graphs of the following functions. Clearly label axes, intercepts, local max/min.

 (a) $f(x) = \tan(\pi x)$
 (b) $g(s) = 2\sin(3s)$
 (c) $h(t) = 1 - \cos t$
 (d) $h(t) = \sec t$
 (e) $f(x) = x^3 - x$
 (f) $g(x) = x^2 + 4x + 2$
 (g) $h(x) = x(x+1)^2(2-x)^3$

27. Let $f(x) = \dfrac{4-x}{3+x}$.
 (a) Find the intercepts of f and all its asymptotes. Also find the limiting behaviour of f near its vertical asymptotes ($\lim_{x\to a^\pm} f(x)$).
 (b) Use the above information to sketch a guess for the graph of f.
 (c) Confirm details of your guess by finding the intervals where the function is increasing/decreasing and concave up/down.
 (d) Use algebra (long division in this case) to rewrite the function as a polynomial plus a proper quotient. Now sketch a graph of f using translations/dilations of $y = \frac{1}{x}$. Which way of obtaining the sketch do you prefer?
28. Find all roots (if possible), asymptotes, intervals of increase/decrease, intervals of concave up/down to sketch a clear graph of the given function on their domain. Use symmetry when possible.

 (a) $f(x) = 10 + 27x - x^3$
 (b) $f(x) = x^2 - x^4$
 (c) $f(x) = x + \dfrac{2}{x}$
 (d) $f(x) = \dfrac{1}{1+x^2}$
 (e) $f(x) = \dfrac{x}{1+x^2}$

29. The reaction rate V of a common enzyme reaction is given in terms of substrate level S by

$$V = \frac{V_* S}{K + S}, \quad S \geq 0$$

where V_* and K are positive constants.
 (a) Show that V is a strictly increasing function of S. Explain why it follows that V has no absolute maximum value.
 (b) What is $\lim_{S\to\infty} V$?
 (c) Determine the concavity of the graph of V (where is it concave up? where concave down?).
 (d) Sketch a graph of V as a function of S.

Rates of change.

30. §2.7: 30 (vibrating string)
31. If $Q = Q(p)$ is the quantity (in pounds) of a ground coffee that is sold by a coffee compay at a price of p dollars per pound,
 (i) What is the meaning of $Q'(8)$? What are its units?
 (ii) Is $Q'(8)$ positive or negative? Explain.
32. If the units of x are feet and the units for $a(x)$ are pounds per foot, what are the units for da/dx? what units does $\int_2^8 a(x)\,dx$ have?
33. A particle moves along a straight line with position function $s(t) = t^3 - 9t^2 + 15t + 10$, $t \geq 0$, where t is measured in seconds and s is measured in feet.
 (a) Find the velocity at time t.

(b) When is the particle at rest?
(c) When is the particle moving in the negative direction?
(d) When is the particle moving in the positive direction?
(e) Find the total distance traveled during the first 8 s. (It may help to draw a diagram that illustrates the motion of the particle as in Figure 2, page 174.)

Related Rates. Given a relation between quantities, find a relation between their derivatives.

34. §2.8: 1-14

Linearization. Find the linearization $L(x)$ of a function $f(x)$ at a base point $x = a$. Use it to approximate $f(x) \approx L(x)$. Approximate the change of a function Δf near a base point $x = a$ by the change in its linearization, $\Delta L = f'(a)\Delta x$, when x changes by Δx. Equivalently, $\Delta f \approx f'(a)\Delta x$. Or using differentials, $df = f'(a)\,dx$ approximates the change in f when x changes from a by dx.

35. Find the linear approximation of $f(x) = (1+x)^k$ at $x = 0$, where k is any real number. Use your result to approximate $\sqrt{0.9}$.
36. Use linearization to approximate the volume of a cylindrical shell of average radius r, height h, and thickness Δr.
37. Use linear approximations to estimate the amount of paint needed to apply a coat of paint 0.05cm thick to a hemispherical dome with diameter 50 m.
38. The dosage D of diphenhydramine for a dog of body mass w kg, is $D = kw^{2/3}$ mg, where k is a constant. A cocker spaniel has mass $w = 10$ kg according to a veterinarians scale. Use linear approximations to estimate the maximum allowable error Δw in w if the percentage error $\Delta D/D$ in the dosage D must be less than 5%.

Optimization.
Find absolute maxima/minima for continuous functions on bounded domains.
Find all absolute and local maxima/minima for any given function. Justify your answer (either by graph, or using first derivative test or second derivative test).

39. Find the local and absolute maximum and minimum values of the function on the given interval.

(a) $f(x) = x - \sqrt{x}, \quad [0,4]$
(b) $f(x) = \dfrac{x}{x^2+x+1}, \quad [-2,1]$
(c) $f(x) = x - \sqrt{2}\sin x, \quad [0,\pi]$
(d) $f(x) = \cos^2 x - 2\sin x, \quad [0,2\pi]$

40. The velocity of a wave of length L in deep water is

$$v = K\sqrt{\frac{L}{C} + \frac{C}{L}}$$

where K and C are positive constants. What is the length of the wave that gives the minimum velocity?
41. §3.7: 9, 19, 20, 45, 51
42. Assume that if the price of a certain book is p dollars, then it will sell x copies where $x = 7000(1 - p/35)$. Suppose the cost of producing those x copies is $15000 + 2.5x$ dollars. Finally, assume that the company will not sell this book for more than \$35. Determine the price for the book that will maximize profit.

Differential equations and Initial Value Problems. Solve first and second order differential equations of the form $y' = f(x)$ or $y'' = f(x)$ with or without initial conditions.

43. Solve the differential equation $f'(x) = 8x - 3\sec^2 x$

44. Solve the initial value problem $f'(u) = \dfrac{u^2+\sqrt{u}}{u}$, $\quad f(1) = 3$.
45. Find $f(x)$ if $f''(x) = \sin x + \cos x$, $f(0) = 3$, $f'(0) = 4$.

D. Applications of the definite integral.

Total change.

46. §4.4: 70 (given graph of rate of change of volume of water in tank, find amount of water)
47. A particle moves on a line with velocity $v(t) = 3t^2 - 3t - 6$, $t \geq 0$.
    (a) Find the distance that the particle travels in the time interval $0 \leq t \leq 3$.
    (b) Find the particle's total displacement in the time interval $0 \leq t \leq 3$.
48. A rabbit population starts with 80 rabbits and increases at a rate of dP/dt rabbits per week. What does
$$80 + \int_0^{12} \frac{dP}{dt}\, dt$$
represent?

Areas

49. Use geometry to evaluate $\int_0^2 (3x - 2\sqrt{4-x^2})\, dx$.
50. Sketch the region enclosed by the given curves. Set up and evaluate an integral to find the area of the region.
    (a) $y = x^2 + 2$, $y = -x - 1$, $x = 0$, $x = 1$.
    (b) $y = x^2 - 4x$, $y = 2x$.
    (c) $x = 1 - y^2$, $x = y^2 - 1$.

Solids of revolution

51. Find the volumes of the solids obtained by rotating the region bounded by the curves $y = x$ and $y = x^2$ about the following lines:
    (a) the x-axis (b) the y-axis (c) $y = 2$
52. (a) Use calculus to find the volume of a sphere of radius R.
    (b) Find the volume of the sphere after a hole of radius $R/2$ has been drilled through its center.
    (c) What percentage of the volume of the whole sphere is left after drilling the hole?

E. Combination

53. For the function $f(x) = |x^2 + x|$,
    (a) Sketch the graph of $g(x) = x^2 + x$.
    (b) Sketch the graph of f.
    (c) Use your graph above to sketch the graph of $f'(x)$
    (d) Find a formula for $f'(x)$. Confirm that it is consistent with your graph in (c).
    (e) Is f continuous everywhere? Explain.
    (f) Is f differentiable everywhere? Explain.
    (g) Find $\int_{-1}^{2} f(x)\, dx$
54. §4.3: 4