Math 1522 – Final Exam Review – Fall 2024

Topics and Sample Problems

A. Fundamentals

Graphing. *All graphs should clearly show domains, intercepts, limiting behaviour, local max/min, and be clearly labeled.*

1. Know the graphs of $\sin(x), \cos(x)$. From these graphs you can obtain the graphs of $\tan(x)$, $\sec(x)$, $\sin^{-1}(x)$, $\cos^{-1}(x)$, $\tan^{-1}(x)$.
2. Know the graphs of $\ln(x)$, e^x.
3. More generally, know how to obtain the graph of $f^{-1}(x)$ (if it exists) from the graph of $f(x)$.
4. Know the definitions $\sinh(x)$, $\cosh(x)$, $\tanh(x)$, $\operatorname{sech}(x)$, and use them to obtain their graphs.
5. Sketch the graphs of basic functions and their translations using roots, symmetry, behaviour at infinity, asymptotes, superposition, etc.

   (a) $f(x) = x^3 - x$ (b) $x^2 + ax + b$ (c) $x(x-1)^2(x+2)^3$
   (d) $f(x) = 1/x$ (e) $f(x) = 1/x^2$ (f) $f(x) = x + 1/x$
   (g) $f(x) = 1 - e^x$ (h) $f(x) = \ln(x - c)$ (i) $f(x) = a\sin(x - c)$
   (j) $f^{-1}(x)$, given that $f(x) = \sqrt{x-1}$

Derivatives and Limits. *Find derivatives using product rule, quotient rule, chain rule, implicit differentiation, and logarithmic differentiation. Find limits as* $x \to \pm\infty$ *of rational functions, exponentials, logarithms. Use L'Hôpital's rule when appropriate.*

6. Know or be able to find the derivatives of x^p, e^x, a^x, $\ln x$, $\log_a x$, x^x, $\sin x$, $\cos x$, $\tan x$, $\sec x$, $\cot x$, $\csc x$, $\sin^{-1} x$, $\cos^{-1} x$, $\tan^{-1} x$, $\csc^{-1} x$, $\sec^{-1} x$, $\cot^{-1} x$, $\sinh x$, $\cosh x$, $\tanh x$, $\operatorname{sech} x$, $\int_a^x f(t)\,dt$.
7. How are the derivatives of f and f^{-1} related? $(f^{-1})'(a) = \dfrac{1}{f'(f^{-1}(a))}$.
8. Find $(f^{-1})'(3)$, where $f(x) = \sqrt{x-1}$.
9. Find the derivatives of the following functions.

   (a) $f(x) = x2^x$ (b) $f(x) = \dfrac{\tan^{-1} x}{x}$ (c) $f(x) = \sinh(\ln x)$
   (d) $f(x) = \int_1^{\sqrt{x}} \tan^{-1} s\,ds$ (e) $f(x) = x\sin^{-1}(x^3)$

10. Use derivatives and limits to find the graphs of functions, such as
    (a) $f(x) = \tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}}$
    (b) $f(t) = t^2 e^{-kt}$
    (c) $f(x) = x\ln x$
    (d) $f(x) = \dfrac{\sin x}{x}$

B. Integration

Evaluate integrals using substitution (including trigonometric substitution), integration by parts, or partial fractions. Use proper notation for improper integrals.

11. An improper integral $\int_a^\infty f(x)\,dx$ converges if $f(x)$ is continuous and approaches zero sufficiently fast. How fast is fast enough?
12. Evaluate the following definite and indefinite integrals. Some of them may be improper.

    (a) $\displaystyle\int \frac{1 + x - x^2}{x^2}\,dx$ (b) $\displaystyle\int \frac{2^{5x}}{5^{2x}}\,dx$ (c) $\displaystyle\int \frac{4+x}{x^2+4} + \frac{3}{2-3x}\,dx$

(d) $\displaystyle\int \frac{1}{x(2+x^2)}\,dx$ (e) $\displaystyle\int_{-\infty}^{2} x^2\,dx$ (f) $\displaystyle\int_{100}^{\infty} \frac{1}{\sqrt{x}}\,dx$

(g) $\displaystyle\int_{100}^{\infty} \frac{1}{x^2}\,dx$ (h) $\displaystyle\int_{0}^{1} \frac{dx}{x}$ (i) $\displaystyle\int_{5}^{\infty} xe^{-x}\,dx$

(j) $\displaystyle\int_{0}^{1} 9z^2 \ln z\,dz$ (k) $\displaystyle\int_{0}^{\pi/2} \sin^3\theta\cos^2\theta\,d\theta$ (l) $\displaystyle\int \cos^2\theta\,d\theta$

(m) $\displaystyle\int \frac{\cos x}{1+\sin^2 x}\,dx$ (n) $\displaystyle\int_{0}^{a} x^2\sqrt{a^2-x^2}\,dx$ (o) $\displaystyle\int \frac{2+x}{1-x^2}\,dx$

(p) $\displaystyle\int \frac{dv}{v^2+2v-3}$

13. Write the form of the partial fraction expansion of the following terms *without* solving for the unknown constants. (For example $\frac{1}{x^2-1} = \frac{A}{x-1} + \frac{B}{x+1}$.)
    (a) $\dfrac{3x^2-4x+5}{(x-1)(x^2+1)}$
    (b) $\dfrac{1}{(2x-1)^2(x-2)(x^2+4)}$

C. Series $\sum_{n=n_0}^{\infty} a_n$

Know the difference between a sequence and a series. What does it mean for a sequence to converge? What does it mean for a series to converge? What is the sum of a series? What are partial sums? Know p-series and geometric series. Be able to recognize a telescoping series and determine its sum if it converges. Know the divergence test, integral test, direct comparison test, limit comparison test, alternating series test, and ratio test. Know the alternating series estimation theorem. What does it mean for a series to converge absolutely? Conditionally?

14. A series $\sum_{n=n_0}^{\infty} a_n$ converges if the terms a_n approach zero sufficiently fast! How fast is fast enough?
15. Suppose $\sum_{n=4}^{\infty} a_n = 3$. Write a formula for the partial sum s_n of the series. What is $\lim_{n\to\infty} a_n$? What is $\lim_{n\to\infty} s_n$?
16. Define what it means for a series $\sum_{k=k_0}^{\infty} a_k$ to converge absolutely, and conditionally.
17. What is a p-series? Which p-series converge, which diverge?
    What is a geometric series? Which geometric series converge, which diverge?
18. State a formula for
    (a) $S_n = \sum_{k=0}^{n} r^k$ (any value of r, and any integer $n > 0$). Can you derive this formula?
    (b) $S = \sum_{k=0}^{\infty} r^k$, $|r| < 1$.
19. State the integral test. Clearly state all conditions and conclusions.
20. Determine whether the following series converge absolutely, conditionally, or diverge.

(a) $\displaystyle\sum_{n=0}^{\infty} \frac{1+n^3}{1+2n^3}$ (b) $\displaystyle\sum_{n=10}^{\infty} (-1)^n \frac{1+n}{1+2n^3}$ (c) $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{1}{\sqrt{n}}$

(d) $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{\sqrt{n}}{n+1}$ (e) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$ (f) $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{n!}{3^n}$

21. Determine whether the following series converge, and if so, find the sum.

(a) $\displaystyle\sum_{n=0}^{\infty} \frac{2(-1)^n 3^{n+1}}{5^n}$ (b) $\displaystyle\sum_{n=2}^{\infty} \frac{2(-1)^n 3^{n+1}}{5^n}$ (c) $\displaystyle\sum_{n=1}^{\infty} (\ln(n+1) - \ln n)$

(d) $\displaystyle\sum_{n=1}^{\infty} e^{-2n}$ (e) $\displaystyle\sum_{n=2}^{\infty} \frac{-2}{n(n+1)}$ (f) $\left(\frac{\pi}{4}\right)^2 + \left(\frac{\pi}{4}\right)^3 + \left(\frac{\pi}{4}\right)^4 + \cdots$

D. Power series. Taylor Series.

What is a power series in a variable x? What do you know about the region of convergence of a power series (the set of values of x where the series converges)? How do you find it? Where does a power series converge absolutely?

When can you differentiate or integrate power series term-by-term? Does the interval of convergence change?

22. Consider the power series $f(x) = \displaystyle\sum_{n=1}^{\infty} \frac{x^n}{n^2}$.
    (a) Find the interval and radius of convergence for the series representation of f.
    (b) Find series representations for f', f'', and their intervals and radius of convergence.
23. Several important functions that arise in the mathematical sciences are given in terms of power series. An example is the Bessel function of order 0, given by

$$J_0(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n}(n!)^2}$$

Find the interval of convergence of the series for $J_0(x)$.

Taylor series. *A Taylor series of a function $f(x)$ about a basepoint $x = a$ is the power series*

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \cdots .$$

- *Know the Maclaurin series (Taylor series about $x = 0$) for $e^x, \cos x, \sin x, 1/(1-x)$ and their intervals of convergence.*
- *To find Maclaurin series, whenever possible, use these known series. That is, obtain new series by substitution, integration, differentiation, and multiplication by powers of x of known series.*
- *Approximate a function by a Taylor polynomial and estimate the error using Taylor's Inequality, or by the Alternating Series Estimation Theorem, when applicable.*
- *Use Maclaurin series to obtain simplified approximation formulas in physics and engineering applications that involve a small parameter.*

24. Find the Maclaurin series for the following functions and state their radius and interval of convergence:

(a) $f(x) = \sin(x^3)$ (b) $f(t) = \dfrac{1}{1+2t^2}$ (c) $f(x) = xe^x$

(d) $F(x) = \displaystyle\int_0^x \frac{\sin t}{t}\, dt.$ (e) $g(t) = \dfrac{1}{(1+t)^2}$

25. Let $f(x) = \sin(x^3)$. Find $p_9(x)$, the 9th degree Taylor polynomial for f centered at $x = 0$. If $p_9(x)$ is used to approximate $f(x)$, find an upper bound for the error $|p_9(x) - f(x)|$ for $-.1 \le x \le .1$. (Hint: use the Alternating Series Estimation Theorem.)
26. Let $f(x) = 1/x$. Find $p_2(x)$, the 2nd degree Taylor polynomial for f centered at $x = 1$. If $p_2(x)$ is used to approximate $f(x)$, find an upper bound for the error $|p_2(x) - f(x)|$ for $.7 \le x \le 1.3$. (Hint: use Taylor's Inequality.)
27. Approximate $\displaystyle\int_0^{0.5} x^2 e^{-x^2}\, dx$ to within $|error| < 0.01$.
28. Approximate $\ln(1.1)$ to within $|error| < 10^{-4}$.
Hint: start by finding the Maclaurin series for $\ln(1 + x)$.
29. Use series to evaluate the limit $\displaystyle\lim_{x\to 0} \frac{1 - \cos x}{1 + x - e^x}$
30. Show that $\cosh x \ge 1 + \frac{1}{2}x^2$ for all x. *Hint: use Maclaurin series*
31. Use known Maclaurin series to find the following sums
(a) $\displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{3^n}{n!}$ (b) $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{3^n}{n!}$ (c) $\displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{3^{2n+1}}{(2n+1)!}$
(d) $\pi - \pi^3/3! + \pi^5/5! - \pi^7/7! + \cdots$
32. When a voltage V is applied to a series circuit consisting of a resistor R and an inductor L, the current at time t is
$$I = \left(\frac{V}{R}\right)(1 - e^{-Rt/L}) \ .$$
Use Taylor series to deduce that $I \approx Vt/L$ if R is small.

Complex numbers. *Convert a complex number from* Cartesian form $z = a + ib$ *to* exponential form $z = re^{i\theta}$, *and vice-versa. Add, subtract, multiply, and divide complex numbers. Compute the complex conjugate and the modulus of a complex number. Raise complex numbers to powers. Be able to write the final answer in the form* $a + ib$ *or* $re^{i\theta}$. *Plot complex numbers in the complex plane. Know and be able to use Euler's Formula:* $e^{i\theta} = \cos\theta + i\sin\theta$.

33. Homework 11: problems 7-13

E. Differential Equations

Check whether a given function solves a given differential equation or initial value problem. Determine when a solution is increasing or decreasing without knowing the solution. Find equilibrium solutions. Understand direction fields. Solve separable equations and initial value problems. Identify and solve linear equations using the method of integrating factors.

34. Consider the differential equation
$$\frac{dI}{dt} = 15 - 3I$$
(a) For which values of I is $I(t)$ increasing? decreasing? Constant?
(b) Find the solution $I(t)$ if $I(0) = 3$.
35. Let $P(t)$ be the population at time t. Suppose $P(t)$ satisfies the differential equation
$$\frac{dP}{dt} = rP(1 - \frac{P}{M})$$
where r and M are positive constants. For which values of P is $P(t)$ increasing? Decreasing? Constant?

36. §9.2: 1, 3-10
37. Find all solutions to the differential equations (these should be easy).
    (a) $y' = t$ (b) $y' = y$ (c) $\mu' = 2\mu$
38. Solve the following initial value problems.
    (a) $\dfrac{dy}{dx} = e^x(1 + 2y)\ , \quad y(0) = 2\ .$
    (b) $\dfrac{dy}{dt} = t^2 y\ , \quad y(1) = -1\ .$
    (c) $\dfrac{dy}{dt} = ty^2\ , \quad y(1) = -1\ .$
39. Find the general solution of the differential equation: $2ty' + y = 4t\ , \quad t > 0.$