Math 1522 - Exam 3 Review - Fall 2024

Exam 3 covers §11.1-11.11, Complex Numbers, HW 7-11 (unless told otherwise by your instructor)

Topics:

1. **Sequences**: Use patterns to find a formula for the nth term of a sequence, determine whether a sequence converges or diverges, find the limit of a sequence (if it exists), use limit laws and/or squeeze theorem where applicable.

2. **Series**:
   - Know the meaning of convergence/divergence of a series in terms of a limit of partial sums.
   - Find the sum of a converging Telescoping or Geometric Series.
   - Understand absolute and conditional convergence, give examples.
   - Determine whether a series converges absolutely, converges conditionally, or diverges using (possibly a combination of): the definition of convergence of a series, Telescoping Series, Geometric Series, Divergence Test, Integral Test, p-series, Direct Comparison Test, Limit Comparison Test, Alternating Series Test, Ratio Test.
   - Estimate the sum of a converging Alternating Series using partial sums, understand and use the Alternating Series Estimation Theorem (for example, provide an upper bound for the error or determine how many terms are needed to approximate a given series to within a given max error).

3. **Power Series**: $\sum_{n=0}^{\infty} c_n(x-a)^n$
   - Understand and be able to find the radius and interval of convergence of a power series.
   - Find a power series for a function $f(x)$ using geometric series and algebraic manipulation, substitution, multiplication/division by constants and powers of $x-a$, or term by term differentiation or integration.

4. **Taylor Series**:
   - Find the Taylor series of a function $f(x)$ centered at a
     - using the definition $\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$
     - or using known Taylor series and algebraic manipulation, substitution, multiplication/division by constants and powers of $x-a$, or term by term differentiation or integration.
   - Know the Maclaurin series (Taylor series centered at $x=0$) for

$$\frac{1}{1-x}, \quad e^x, \quad \cos x, \quad \sin x.$$

5. **Taylor Polynomials and Approximation of Functions**:
   - Given a function $f(x)$, find the nth-degree Taylor polynomial $p_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k$.
   - When approximating $f(x)$ by $p_n(x)$ on some interval, give an upper bound for the error: $|f(x) - p_n(x)|$
     - using Taylor's Inequality
     - or using the Alternating Series Estimation Theorem (if applicable).

6. **Applications**: Use Taylor Series or Taylor Polynomials to
   - Compute limits.
   - Approximate definite integrals.
   - Approximate functions by polynomials.
   - Obtain simplified approximation formulas in physics and engineering applications that involve a small parameter.

7. **Complex Numbers and Euler's Formula**
   - Convert a complex number from *Cartesian form* $z = a + ib$ to *exponential form* $z = re^{i\theta}$, and vice-versa.
   - Add, subtract, multiply, and divide complex numbers. Raise complex numbers to powers. Be able to write the final answer in the form $a + ib$ or $re^{i\theta}$.
   - Plot complex numbers in the complex plane.
   - Know and be able to use Euler's Formula: $e^{i\theta} = \cos\theta + i\sin\theta$.

Sample Problems:

1. What does it mean to say that $\lim_{k\to\infty} a_k = L$?
2. What does it mean to say that $\sum_{k=1}^{\infty} a_k = L$?
3. Assume $\sum_{k=1}^{\infty} a_k = L$. What is $\lim_{k\to\infty} a_k$? What is $\lim_{n\to\infty} s_n$, where $s_n = \sum_{k=1}^{n} a_k$?
4. Let $a_k = 1/2^k$.
   (a) Sketch a plot of the sequence $\{a_k\}$.
   (b) Sketch a plot of the sequence $\{s_n\}$, where $s_n = \sum_{k=1}^{n} a_k$?
   (c) Find $\sum_{k=1}^{\infty} a_k$.
5. What is a Telescoping Series? Give an example of a Telescoping Series and determine its sum, or determine that it diverges. *Note: To determine the sum of a Telescoping Series, first find a formula for its partial sums, and then take their limit.*
6. Consider the Geometric Series $\sum_{k=k_1}^{\infty} r^k$. For which r values does the series converge? Diverge? Give two specific examples of converging Geometric Series, one starting at $k_1 = 0$, another starting at $k_1 \neq 0$. Find the sum of each.
7. For which values of p does the p-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converge? Diverge?
8. What is an absolutely converging series? What is a conditionally converging series? Give an example of each.

9. Fill in the blanks using either *may* or *must*. Briefly explain your answers.
   (a) A series with terms tending to 0 __________ converge.
   (b) A series that converges __________ have terms that tend to zero.
   (c) If a series diverges, then the terms __________ not tend to 0.
   (d) If a series diverges, then the Divergence Test ________ succeed in proving the divergence.
   (e) If $\sum_{n=10}^{\infty} a_n$ diverges, then $\sum_{n=1000}^{\infty} a_n$ __________ diverge.
   (f) If $\sum_{n=10}^{\infty} a_n$ converges, then $\sum_{n=10}^{\infty} |a_n|$ __________ converge.
   (g) If $\sum_{n=10}^{\infty} |a_n|$ converges, then $\sum_{n=10}^{\infty} a_n$ __________ converge.
10. Chapter 11 Review, True-False: 1-3,7-12,14-22
11. Chapter 11 Review, Exercises: 1-8 (Determine whether the sequence converges or diverges, and find the limit if it converges)
12. Chapter 11 Review, Exercises: 11-22 (Determine whether the series converge or diverge)
13. Chapter 11 Review, Exercises: 23-26 (Determine whether the series converges absolutely, conditionally, or diverges)
14. Chapter 11 Review, Exercises: 27-29 (Find the sum of the series)
15. Show that the series $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k^3}$ converges. How many terms are needed to approximate the sum with error less than 10^{-3}?
16. Chapter 11 Review, Exercises: 38
17. Determine the radius and interval of convergence of the power series $\displaystyle\sum_{n=1}^{\infty} \frac{3^n (x+4)^n}{\sqrt{n}}$.
18. Chapter 11 Review, Exercises: 40-43 (Find the radius and interval of convergence of the power series)
19. Chapter 11 Review, Concept Check: 9, 11
20. Let $f(x) = \displaystyle\sum_{n=1}^{\infty} \frac{x^n}{n^2}$. Find series representations for f', f''. Find the intervals of convergence of the series for f, f', and f''.
21. Chapter 11 Review, True-False Quiz: 4-6, 10, 13
22. Use series to show that $\cosh x > 1 + \frac{1}{2}x^2$ for all x.
23. Let $f(x) = \sin x$.
   (a) Find the first 5 nonzero terms of the Taylor series for f, centered at $a = \pi/6$.
   (b) Give $p_2(x)$, the Taylor polynomial of degree 2 for f, centered at $a = \pi/6$.
   (c) Find an upper bound for the error $|f(x) - p_2(x)|$ on the interval $[0, \pi/3]$.
24. Find power series for the following functions and give their radius and interval of convergence. (see also Chapter 11 Review, Exercises: 47-51.)
   (a) $f(x) = \dfrac{2}{3-x}$
   (b) $f(x) = \dfrac{x}{2x^2+1}$
   (c) $f(x) = \tan^{-1}(x)$
   (d) $f(x) = \dfrac{x}{(2-x)^2}$
   (e) $f(x) = \displaystyle\int_0^x \frac{t}{1-t^8}\, dt$
   (f) $f(x) = x\sin(x^2)$

25. Find the power series for $\ln(1+x^2)$ about $x=0$. Use the resulting series to estimate the value of $\ln(1.01)$ to within 10^{-7}.

26. Use series to approximate the integrals

    (a) $\displaystyle\int_0^{0.1} \frac{1}{1+x^4}\,dx$ to within 10^{-7}.

    (b) $\displaystyle\int_0^{1} \frac{\sin x}{x}\,dx$ to within 10^{-2}.

27. Use series to evaluate the limits.

    (a) $\displaystyle\lim_{x\to 0} \frac{\sin x - x + \frac{1}{6}x^3}{x^5}$

    (b) $\displaystyle\lim_{x\to 0} \frac{\sqrt{1+x} - 1 - \frac{1}{2}x}{x^2}$

28. (a) Find the function represented by the power series $\displaystyle\sum_{n=2}^{\infty} n(n-1)x^n$ and find its interval of convergence.

    (b) Evaluate the sum of the series: $\displaystyle\sum_{n=2}^{\infty} \frac{n(n-1)}{2^n}$.

29. Let $f(x) = \sqrt[3]{x}$.

    (a) Find $p_2(x)$, the 2nd-degree Taylor polynomial for f centered at $a=8$.

    (b) Find an upper bound for the error $|f(x) - p_2(x)|$ for $7 \le x \le 9$.

30. When a voltage V is applied to a series circuit consisting of a resistor R and an inductor L, the current at time t is

$$I = \left(\frac{V}{R}\right)\left(1 - e^{-Rt/L}\right) .$$

    (a) Find the Taylor series for I centered at $R=0$

    (b) Deduce that $I \approx Vt/L$ if R is small.

31. Complex Numbers: HW11 problems 7-13.