HOMEWORK DAY 29 – *Taylor series §11.10*

Note on Uniqueness: The power series representation $\sum_{n=0}^{\infty} c_n(x-a)^n$ of a function about the base point $x = a$ is unique: if f has a power series representation about $x = a$ that converges to f in $(a - R, a + R)$, for some $R > 0$, then the unique coefficients are given by $c_n = \frac{f^{(n)}(a)}{n!}$

Consequence: If a function has a power series representation about the basepoint $x = a$, then the power series is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \dots$$

This (formal) series is called the Taylor series. If the Taylor series converges, it equals the *power series representation of f.*

1. 11:10: 1

2. 11:10: 2

3. If $f(x) = 2x - x^2 + \frac{1}{3}x^3 - 5x^4 - x^6/2 + 2x^7 + \dots$ converges for all x, what is $f^{(7)}(0)$? Can you find it without differentiating $f(x)$?

4. Find the Taylor series for f about $x = 4$ if

$$f^{(n)}(4) = \frac{(-1)^n n!}{3^n(n+1)}$$

What is its radius of convergence?

5. Find the MacLaurin series for $f(x) = \cosh(x)$.

6. Find the first 3 terms of the Taylor series for $f(x) = \sqrt{x}$ centered at $x = 4$.

7. Find the Taylor series for $f(x) = x - x^3$ about $x = 2$.

8. Use the (memorized) Taylor series about $x = 0$ (ie, McLaurin series) for

$$\frac{1}{1-x}\ , \quad \cos x\ , \quad \sin x\ , \quad e^x$$

to find the Mclaurin series for

(a) $f(x) = x\cos(2x)$

(b) $f(x) = \tan^{-1}(x^4)$ (Hint: First find the series for $\tan^{-1} x$)

(c) $f(x) = x\ln(1 + x^3)$ (Hint: First find the series for $\ln(1 + x)$)

9. Use series to evaluate the limit

   (a) §11.10: 67

   (b) §11.10: 68

   (c) §11.10: 69

   (d) §11.10: 70

10. *Application: approximating definite integrals.* Estimate the value of the following integral

$$\int_0^{0.2} \frac{dx}{1+x^4}$$

to six decimal places.

11. Find a function represented by the following power series

(a) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^{4n}}{n!}$

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n 2^{2n} x^{2n}}{(2n)!}$

(c) $\displaystyle\sum_{n=0}^{\infty} n x^{n-1}$

(d) $\displaystyle\sum_{n=0}^{\infty} n x^{n}$

(e) $\displaystyle\sum_{n=0}^{\infty} n(n-1) x^{n}$

12. Find the sum of the series

(a) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{n!}$

(b) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^{n+1}\pi^{2n}}{2^{2n}(2n)!}$

(c) $\displaystyle\sum_{n=2}^{\infty} \frac{5n(n-1)}{3^n}$

13. (a) Write down a formula for the Taylor series of $f(x)$ about $x = a$.

(b) Write down a formula for the Taylor polynomial $p_n(x)$ of degree n of a function $f(x)$ about a basepoint $x = a$.

(c) If p_n is used to approximate a function f about $x = a$, give an upper bound for the error $|f(x) - p_n(x)|$. (I.e., state Taylor's Inequality)

14. Let $f(x) = e^x$.

(a) Find $p_2(x)$ for $f(x)$ about $x = 0$.

(b) Find an upper bound for $|f(x) - p_2(x)|$ if $|x| < 0.1$.

(c) Sketch a graph of f and p_2.

15. §11.11: 13. Replace part (c) by: sketch a graph of f and p_2 over the given interval.

16. Here we address: How good is the approximation $\sin\theta \approx \theta$ if θ is small?

(a) State the Taylor series for $f(\theta) = \sin\theta$ about $\theta = 0$.

(b) Find the linear approximation $p_1(\theta)$.

(c) Find an upper bound for $|f(\theta) - p_1(\theta)|$ if $|\theta| \leq 0.1$ using the Alternating Series Remainder Theorem.

(d) Find an upper bound for $|f(\theta) - p_1(\theta)|$ if $|\theta| \leq 0.1$ using Taylor's Inequality.