HOMEWORK DAY 32 – *Applications of Taylor series to physics and Engineering §11.11*

1. §11.11: 32(a)

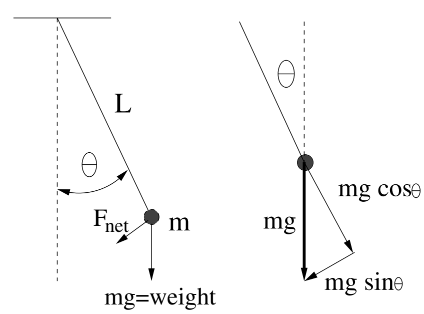
2. *A simple pendulum.* An idealized simple pendulum is given by a mass m hanging from a massless string of length L. Its motion is described by the angle $\theta(t)$, where t is time. The distance travelled by the pendulum is the arclength $s(t) = L\theta(t)$. According to Newton's 2nd law, the pendulum mass×acceleration equals the restoring force F_{net} acting on it.

$$mL\theta''(t) = F_{net}$$

where $F_{net} = mg \sin\theta$ (see picture). For small angles one often uses the approximation $\sin\theta \approx \theta$ to replace this differential equation by the simpler and easily solvable equation

$$mL\theta''(t) = mg\theta$$

Question: Consider the approximation $\sin\theta \approx \theta$. If the angle swings with $-\pi/10 \leq \theta \leq \pi/10$, what is an upper bound for the error made in this approximation?



3. Find the Taylor series for $f(x) = \dfrac{1}{(1+x)^2}$. (Hint: differentiate the geometric series.)

4. Use the above results in the following problem.
An electric dipole consists of two electric charges of equal magnitude and opposite signs. If the charges are q and $-q$ and are located at a distance d from each other, then the electric field E at the point P in the figure is

$$E = \frac{q}{D^2} - \frac{q}{(D+d)^2}$$

By expanding this expression for E as a series in powers of d/D, show that E is approximately proportional to $1/D^3$ when P is far away from the dipole (that is, when D is much bigger than d, so that $d/D \ll 1$).

5. When a voltage V is applied to a series circuit consisting of a resistor R and an inductor L, the current at time t is $I = \left(\frac{V}{R}\right)\left(1 - e^{-Rt/L}\right)$. Use Taylor series to deduce that $I \approx Vt/L$ if R is small.

6. §11.11: 35

7. Evaluate the expression and write your answer in the form $a + bi$

   (a) $(5 - 6i) + (3 + 2i)$

   (b) $(5 - 6i)(3 + 2i)$

   (c) $(1 - 2i)(8 - 3i)$

   (d) $\dfrac{1}{1 + i}$

   (e) $\dfrac{5 - 6i}{3 + 2i}$

(f) i^3

(g) i^{151}

(h) $\sqrt{-25}$

(i) $|5-6i|$

(j) $|-1+2\sqrt{2}i|$

(k) $\overline{5-6i}$

(l) $\overline{-1+2\sqrt{2}}$

(m) $\overline{\left(\dfrac{1}{1+i}\right)}$

8. Find all solutions of the equation. Write your answer in the form $a+bi$.

(a) $4x^2+9=0$

(b) $x^2+2x+5=0$

(c) $2x^2-2x+1=0$

9. Let $z = x + iy$ be a complex number. Define e^z by extending the Maclaurin series for e^x to the complex plane:

$$e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \frac{z^4}{4!} + \cdots$$

Use this definition to prove **Euler's Formula** for any real number θ:

$$e^{i\theta} = \cos\theta + i\sin\theta.$$

It follows that $z = x + iy = re^{i\theta}$ where $x = r\cos\theta$ and $y = r\sin\theta$. That is, r and θ are the polar coordinates of z ($r = |z|$ and $\tan\theta = x/y$). This gives a **Cartesian representation** of $z = x + iy$ and an **exponential representation** $z = re^{i\theta}$.

10. Use Euler's formula to show that:

(a) $\cos\theta = \dfrac{e^{i\theta} + e^{-i\theta}}{2}$

(b) $\sin\theta = \dfrac{e^{i\theta} - e^{-i\theta}}{2i}$

(c) If $z = re^{i\theta}$, then $|z| = r$.

(d) If $z = re^{i\theta}$, then $\overline{z} = re^{-i\theta}$.

(e) $|e^{i\theta}| = 1$

(f) $e^{2n\pi i} = 1$ for any integer n.

(g) $e^{i\pi} = -1$

11. Write $z = e^{2+i\pi/6}$ in Cartesian form $z = a + ib$.

12. Write the following complex numbers in exponential form $z = re^{i\theta}$. Plot each point in the complex plane.

    (a) $-4i$

    (b) $8i$

    (c) $4\sqrt{3} - 4i$

13. Compute the following and write your answer in the form $a + ib$. (Hint: use exponential forms to do the computation, then convert to Cartesian form).

    (a) $(-1+i)^7$

    (b) $(1+i)^{20}$

    (c) $(1-\sqrt{3}i)^5$

    (d) $(1-i)^8$