Math 2531 – Final Exam Review – Fall 2024

Topics and sample problems

I. Basics

Vectors

add and subtract vectors algebraically and graphically

magnitude of vectors, resultant force

Dot and Cross product

magnitude of dot product, magnitude and direction of cross product

what does $\mathbf{a}\cdot\mathbf{b}=0$ *mean? what does* $\mathbf{a}\times\mathbf{b}=\mathbf{0}$ *mean? what is* $\mathbf{a}\cdot\mathbf{a}$*? what is* $\mathbf{a}\times\mathbf{a}$*?*

find scalar and vector projections of $\mathbf{b}$ *onto* $\mathbf{a}$

Lines and Planes

find equations for lines/planes

Graphing

graph basic surfaces $F(x,y,z)=const$ *in cartesian, cylindrical, spherical coordinates*

graph surfaces given by functions $z=f(x,y)$

II. Vector functions $\mathbf{r}(t)=\langle x(t),y(t)\rangle$ or $\mathbf{r}(t)=\langle x(t),y(t),z(t)\rangle$

Sketch elementary curves $\mathbf{r}(t)$ (lines, parabolas, circles, ellipses, helices, etc.)

Compute unit tangent vector, at an arbitrary point $\mathbf{r}(t)$ and at a specific points $\mathbf{r}(t_0)$

Find velocity, speed, acceleration, arc length, curvature

III. Scalar functions $f(x,y)$ or $f(x,y,z)$

Chain Rule

Directional Derivative. Gradient. Level Curves, Level surfaces.

Magnitude and direction of gradient vector, relative to level curves

Tangent Planes

Find Tangent Planes to surfaces $F(x,y,z)=0$ *or* $z=f(x,y)$.

Extrema

Find local and absolute max/min of $f(x,y)$ *on infinite or bounded domains*

IV. Integrals

Double Integrals $\iint_A f(x,y)\,dA$

Evaluate over general regions in cartesian or polar coordinates

Triple Integrals $\iiint_V f(x,y,z)\,dV$

Evaluate over general regions in cartesian, cylindrical or spherical coordinates

Applications : Volume, Mass, Center of Mass.

Line Integral I : $\int_C f(x,y,z)\,ds$

Parametrize C *and evaluate (be able to parametrize circles, ellipses, curves* $y=f(x)$*, lines)*

Line Integral II : $\int_C \mathbf{F}\cdot\mathbf{T}\,ds=\int_C \mathbf{F}\cdot\,d\mathbf{r}=\int_C P\,dx+Q\,dy+R\,dz$

Evaluate when $\mathbf{F}$ *is not conservative (need parametrization of* C*)*

Evaluate when $\mathbf{F}=\nabla f$ *is conservative (don't need parametrization, use fundamental theorem)*

V. Vector fields $\mathbf{F}(x, y) = \langle P(x, y), Q(x, y)\rangle$ or $\mathbf{F}(x, y, z) = \langle P(x, y, z), Q(x, y, z), R(x, y, z)\rangle$

Graph simple vector fields on $\mathbb{R}^2$ or $\mathbb{R}^3$.

Gradient fields $\mathbf{F} = \nabla f$ (i.e., conservative fields)

Given f, find its gradient field ∇f

Given $\mathbf{F}$, determine if it is a gradient field in an open connected region D. (Method 1: show that $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 0$ or $\nabla \times \mathbf{F} = \mathbf{0}$ in D. Method 2: find the potential function directly)

If $\mathbf{F}$ is conservative, find a potential function f (such that $\mathbf{F} = \nabla f$).

If $\mathbf{F} = \nabla f$, know relation between vector field and level curves of f. Plot both in one graph.

Find divergence and curl of a vector field.

Given the graph of a vector field $\mathbf{F}$, determine whether $\nabla \cdot \mathbf{F}$ and $\nabla \times \mathbf{F}$ are zero at a given point.

VI. Green's Theorem

For a positively oriented, piecewise-smooth, simple closed curve C in the xy-plane enclosing region D, if P and Q have continuous partial derivatives on an open region containing D, then

$$\int_C P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

Sample Poblems

I. BASICS

1. (a) Copy the vectors in Fig. 1 and use them to draw each of the following vectors: $\mathbf{a} + \mathbf{b}$, $-\frac{1}{2}\mathbf{a}$, $\mathbf{a} - \mathbf{b}$, $2\mathbf{a} + \mathbf{b}$.
   (b) Is $\mathbf{a} \cdot \mathbf{b}$ positive or negative?
   (c) Does $\mathbf{a} \times \mathbf{b}$ point inside or out of the plane.
   (d) Indicate the length $|\mathbf{a} \cdot \mathbf{b}|/|\mathbf{a}|$ in a figure. Also indicate the length $|\mathbf{a} \cdot \mathbf{b}|/|\mathbf{b}|$.
   (e) Indicate the length $|\mathbf{a} \times \mathbf{b}|/|\mathbf{a}|$ in a figure. Also indicate the length $|\mathbf{a} \times \mathbf{b}|/|\mathbf{b}|$.

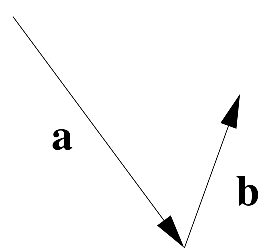


FIG 1

2. (a) If $\mathbf{u}$ and $\mathbf{v}$ are the vectors shown in Fig. 2a, find $\mathbf{u} \cdot \mathbf{v}$ and $|\mathbf{u} \times \mathbf{v}|$. Is $\mathbf{u} \times \mathbf{v}$ directed into the page or out of it?
   (b) Repeat for the vectors shown in Fig. 2b.

3. If $\mathbf{u}$ and $\mathbf{v}$ are the vectors shown in Fig. 3, find $\mathbf{u} \cdot \mathbf{v}$ and $|\mathbf{u} \times \mathbf{v}|$. Is $\mathbf{u} \times \mathbf{v}$ directed into the page or out of it? Find $\mathbf{u} \cdot \mathbf{u}$ and $\mathbf{u} \times \mathbf{u}$.

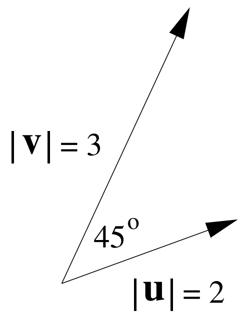


FIG 2a

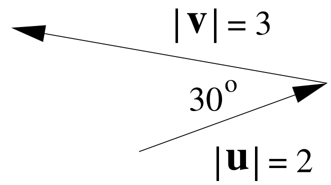


FIG 2b

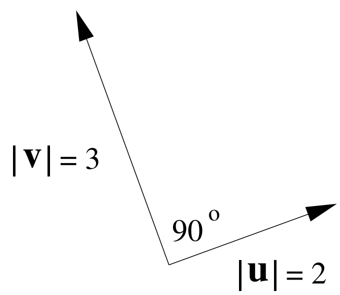


FIG 3

4. Draw two arbitrary vectors $\mathbf{a}$ and $\mathbf{b}$. In the same figure, draw $\text{proj}_{\mathbf{a}}\mathbf{b}$, the vector projection of $\mathbf{b}$ onto $\mathbf{a}$. Write down an expression for the vector projection.

5. How can you use the dot product to determine whether the angle between two vectors is acute or obtuse (less than or greater than 90°)?

6. Given the three points $A(1, 0, 0)$, $B(2, 0, -1)$, $C(1, 4, 3)$. Let $\mathbf{a} = \overrightarrow{AB}$, $\mathbf{b} = \overrightarrow{BC}$, $\mathbf{c} = \overrightarrow{CA}$.
   (a) Find $\text{comp}_{\mathbf{a}}\mathbf{b}$, the scalar projection of $\mathbf{b}$ onto $\mathbf{a}$. Find $\text{proj}_{\mathbf{a}}\mathbf{b}$, the vector projection of $\mathbf{b}$ onto $\mathbf{a}$.
   (b) Find $\text{comp}_{\mathbf{b}}\mathbf{a}$, the scalar projection of $\mathbf{a}$ onto $\mathbf{b}$. Find $\text{proj}_{\mathbf{b}}\mathbf{a}$, the vector projection of $\mathbf{a}$ onto $\mathbf{b}$.
   (c) Find the area of the triangle ABC.
   (d) Find the angle between the vectors $\mathbf{a}$ and $\mathbf{b}$.
   (e) Draw a sketch of the three points A,B,C and the three vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$. From the sketch (ie, without computation) determine $\mathbf{a} + \mathbf{b} + \mathbf{c}$.

7. (a) Find the equation of the plane that passes through the origin and contains the vectors $\mathbf{u} = \langle 0, 1, 2 \rangle$ and $\mathbf{v} = \langle 1, -2, 3 \rangle$.
   (b) Find an equation for the line through the origin that is normal to the plane in part (a).

8. Find the equation of the plane that contains the three points $P(1, 1, 0)$, $Q(0, 2, -1)$, $R(3, 4, 2)$.

9. Find an equation for the plane passing through (-4,1,2) and parallel to the plane $5z = 3 - x - 2y$.

10. Find parametric equations for the line passing through (-6,-1,0) and (2,-3,5).

11. (a) Describe a method to find the distance from a point to a plane. (Be clear and concise, so that someone else can use this method. Include a sketch. Do not refer to a formula in the book.)
   (b) Use your method to find the distance from the origin to the plane $4x - 6y + z = 5$. (Ans: $\frac{5}{\sqrt{53}}$)

12. Determine whether the following pairs of lines are parallel, skew, or intersecting. If they intersect, find the point of intersection. If they are parallel, determine whether they are identical or distinct lines.

    (a) $\mathbf{r}_1(t) = \langle 1, 1, 0\rangle + t\langle 1, -1, 2\rangle$, $\mathbf{r}_2(t) = \langle 2, 0, 4\rangle + t\langle -1, 1, 0\rangle$
    (b) $\mathbf{r}_1(t) = \langle 1, 0, 2\rangle + t\langle 1, 1, -5\rangle$, $\mathbf{r}_2(t) = \langle 1, -2, -1\rangle + t\langle 1, 1, -1\rangle$
    (c) $\mathbf{r}_1(t) = \langle 1, 0, 2\rangle + t\langle 1, 1, -5\rangle$, $\mathbf{r}_2(t) = \langle 1, -2, -1\rangle + t\langle 2, 2, -10\rangle$

13. Find the points in which the line $x = t$, $y = 2 - t$, $z = 2 + t$ intersects the coordinate planes.

14. Describe the graph of the following equations in $\mathbb{R}^3$.

    (a) $x = y^2 + z^2$,
    (b) $z = \sqrt{x^2 + y^2}$,
    (c) $z^2 = x^2 + y^2$
    (d) $x = 4$
    (e) $xy = 4$
    (f) $xy = 0$
    (g) $\theta = 3\pi/4$
    (h) $\phi = 3\pi/4$
    (i) $\phi = 0$
    (j) $\rho = 4$
    (k) $\rho = 4\cos\phi$
    (l) $r\sin\theta = 3$
    (m) $r = 3$
    (n) $r = \cos\theta$
    (o) $x = z$
    (p) $y = 3x - 2$
    (q) $y = z^2$
    (r) $4x^2 + y^2 + z^2 = 2z$
    (s) $y^2 + z^2 = 1 + x^2$
    (t) $6x + 4y + 3z = 12$

15. (a) What is the distance from a point (x, y, z) to the x-axis?
    (b) What is the distance from a point (x, y, z) to the yz-plane?
    (c) Find an equation for the surface consisting of all points (x, y, z) in space for which the distance from (x, y, z) to the x-axis is twice the distance from (x, y, z) to the yz-plane. Simplify the equation, removing square roots and absolute values. Describe and sketch the surface.

II. VECTOR FUNCTIONS

16. Sketch the following curves.

    (a) $\mathbf{r}(t) = \langle \cos 4t, t, \sin 4t\rangle, 0 \le t \le 2\pi$
    (b) $\mathbf{r}(t) = \langle 2\cos t, \frac{1}{2}\sin t, 0\rangle, 0 \le t \le 2\pi$
    (c) $\mathbf{r}(t) = \langle 2 + t, -t, -1 + 2t\rangle$, $0 \le t \le 2$
    (d) $\mathbf{r}(t) = \langle t^4 + 1, t\rangle$, $-\infty \le t \le \infty$
    (e) $\mathbf{r}(t) = \langle t, t, \cos t\rangle$, $-\infty \le t \le \infty$

17. Sketch the given curves, find the unit tangent vector at $\mathbf{r}(t)$, find the unit tangent vector at the indicated point P.

    (a) $\mathbf{r}(t) = \langle \sqrt{2}\sin t, \sqrt{2}\cos t\rangle$, $0 \le t \le \pi$, P(1,1)
    (b) $\mathbf{r}(t) = \langle t^2, t^3\rangle$, $-\infty \le t \le \infty$, P$(1/4, 1/8)$
    (c) $\mathbf{r}(t) = \langle 1 + t, t^2\rangle$, $-\infty \le t \le \infty$, P$(2, 1)$

18. The curves $\mathbf{r}_1(t) = \langle t, t^2, t^3\rangle$ and $\mathbf{r}_2(t) = \langle \sin t, \sin 2t, t\rangle$ intersect at the origin. Find the angle of intersection of the two curves (i.e., the angle between the tangent vectors at the point of intersection).

19. Find the velocity, acceleration and speed of a particle with position $\mathbf{r}(t) = t^2\,\mathbf{i} + \ln t\,\mathbf{j} + t\,\mathbf{k}$.

20. Find the velocity and position vectors of a particle with acceleration $\mathbf{a}(t) = \mathbf{k}$ that one second into the motion has position and velocity $\mathbf{r}(1) = \mathbf{0}$ and $\mathbf{v}(1) = \mathbf{i} - \mathbf{j}$.

21. Find the length of the curve $\mathbf{r}(t) = 2t^{3/2}\,\mathbf{i} + \cos 2t\,\mathbf{j} + \sin 2t\,\mathbf{k}$, $0 \le t \le 1$. (Ans: $\frac{2}{27}(13^{3/2} - 8)$)

22. Find the curvature $\kappa(t)$ of the curve $\mathbf{r}(t) = \langle \cos t, \sin t, t \rangle$.

III. SCALAR FUNCTIONS $f(x, y)$, $f(x, y, z)$

23. Sketch the graphs of the following functions $f(x, y)$. In a separate plot, sketch the level curves.

(a) $f(x, y) = x^2 + 4y^2$
(b) $f(x, y) = \sqrt{x^2 + y^2}$
(c) $f(x, y) = x - 3y$
(d) $f(x, y) = y^2$
(e) $f(x, y) = x^2 - y^2$

24. Sketch the level surfaces of the following functions $f(x, y, z)$.

(a) $f(x, y, z) = x^2 + y^2 + z^2$
(b) $f(x, y) = x + y + 3z$
(c) $f(x, y, z) = x^2 - y^2$

25. Find the rate of change of the function $f(x, y, z) = x^2 + y^2 + z$ along the path $x = \cos t$, $y = \sin t$, $z = t$.

26. If $u = xy + yz + zx$, $x = st$, $y = e^{st}$, $z = t^2$, find $\frac{\partial u}{\partial s}$ and $\frac{\partial u}{\partial t}$ when $s = 0$, $t = 1$.

27. If $f(x, y, z) = \frac{x}{y+z}$, P(4,1,1), $\mathbf{v} = \langle 1, 2, 3 \rangle$,

(a) Find the rate of change of f at P in direction of the vector $\mathbf{v}$.
(b) Find the rate of change of f at P in direction of the x-axis.
(c) In what direction does f increase the fastest at P?
(d) What is the largest rate of increase of f at P?
(e) In what direction does f decrease the fastest at P?
(f) Give a vector normal to the level surface $f = 2$ at P.
(g) Find an equation for the plane tangent to the level surface $f = 2$ at P.

28. Consider the function $f(x, y) = x - y^2$.

(a) Draw the level curves $f(x, y) = k$ for $k = -2, -1, 0, 1, 2$, in the x-y plane. Carefully label the level curves and the axes.
(b) Find the gradient ∇f at (2,1) and enter it in your sketch.
(c) If you stand at the point (2,1) and look toward the origin, does f increase or decrease in that direction?
(d) In what direction does f increase the fastest? What is the rate of maximal increase?

29. §14.6: 26 (Sketch the gradient vector at the points P, Q, and R on the given contour map.)

30. For each of the following surfaces S
(i) write down a function $F = F(x, y, z)$ with the property that S is a level surface of F.
(ii) Write down a vector that is normal to S at the given point
(iii) write down an equation for the tangent plane to S at the given point.

(a) $z = x^2 + y^2$, $P(1, 1, 2)$
(b) $x + y - 2z = 5$, $P(2, 1, -1)$
(c) $xy + xz + yz = 3$, $P(1, 1, 1)$
(d) $x^2 = 2y^2 + 3z^2 - xyz + 4$, $P(3, -2, -1)$

(e) $z = f(x, y)$, where $f(x, y) = xe^y + 3y$ and $(x, y) = (1, 0)$

31. Find and classify all the critical points of

    (a) $f(x, y) = x^2 + y^2 + x^2y + 4$

    (b) $f(x, y) = 2x^3 + xy^2 + 5x^2 + y^2$

32. Find the absolute maximum and minimum values of $f(x, y) = x^2+y^2+x^2y+4$ on the square domain D given by $-1 \le x \le 1$, $-1 \le y \le 1$.

33. Find the point on the plane $x+y+z = 1$ that is closest to the point $(2, 0, -3)$. (Ans: $(8/3, 2/3, -7/3)$)

34. Consider a function $z = f(x, y)$. Estimate the change Δz in the function values if x and y change from a point (x_0, y_0) by an amount Δx and Δy respectively.

35. The wave heights h in the open sea depend on the speed v of the wind and the length of time t that the wind has been blowing at that speed. Values of the function $h = f(v, t)$ are recorded in the table given in Section 14.4, #28.

    (a) Estimate the partial derivatives $\partial h/\partial v$, $\partial h/\partial t$ at the point $(v_0, t_0) = (40, 20)$.

    (b) Estimate the change in the wave height if the wind speed increases from 40 to 43 knots and the length of time t increases from 20 to 24 hours.

    (c) Estimate the wave height if $v = 43$ and $t = 24$.

IV. INTEGRALS

36. (a) Write down a formula for the average f_{avg} of a function of one variable, $f(x)$, on the interval $[a, b]$.

    (b) Find the average of $f(x) = \int_x^1 \cos(t^2)\, dt$ on the interval $[0, 1]$. (Ans: $\frac{1}{2}\sin 1$)

37. (a) The average of the function $f(x, y)$ on the disk $D : x^2 + y^2 \le 2$ is $f_{avg} = 5$. Find the value of the integral $\iint_D f(x, y)\, dA$.

    (b) The density function $\rho(x, y, z)$ for the solid cone $E : x^2 + y^2 \le z^2$, $0 \le z \le 10$ has the average value $\rho_{avg} = 3$ g/cm^3, where x, y, z are measured in cm. Find the mass of the cone, $m = \iiint_E \rho(x, y, z)\, dV$.

38. Evaluate $\iiint_E x\, dV$ where E is the region above $z = \sqrt{x^2 + y^2}$, below $x^2 + y^2 + z^2 = 2$, with $x \ge 0$. (Ans: $\frac{\pi}{4} - \frac{1}{2}$)

39. (a) Set up an integral for the volume of the sphere of radius a centered at the origin, in cartesian coordinates, in cylindrical coordinates, and in spherical coordinates.

    (b) Evaluate the integral using your choice of coordinates.

40. Evaluate $\iiint_E y\, dV$ where E is the region below $z = xy$ and above the triangle in the xy-plane with vertices (1,0), (2,1), and (4,0). (Ans: $\frac{11}{20}$)

41. Evaluate $\iiint_E (x + 2y) dV$ where E is bounded by $y = x^2$, $x = z$, $x = y$ and $z = 0$. (Ans: $\frac{2}{15}$)

42. Set up and evaluate a triple integral that gives the volume inside the cone $z^2 = a^2(x^2 + y^2)$ between the planes $z = 1$ and $z = 2$. (Ans: $\frac{7\pi}{3a^2}$)

43. Evaluate the integral $\int_0^1 \int_{y^2}^1 y \sin(x^2)\, dx\, dy$. *Hint: reverse the order of integration.* (Ans: $\frac{1}{4}(1-\cos 1)$)

44. Find the volume and centroid (center of mass assuming constant density) of the tetrahedron with vertices $(0,0,0)$, $(1,1,0)$, $(0,1,0)$, $(0,1,1)$.
(Ans: Volume $= \frac{1}{6}$, $(\overline{x}, \overline{y}, \overline{z}) = (\frac{1}{4}, \frac{3}{4}, \frac{1}{4})$)

45. (a) Set up an integral for the volume of the region above $z = 1$, below $x^2 + y^2 + z^2 = 4$, with $y \geq 0$ in cartesian, cylindrical and spherical coordinates.
    (b) Evaluate the integral using your choice of coordinates. (Ans: $\frac{5\pi}{6}$)

46. (a) Use cylindrical coordinates to evaluate $\displaystyle\int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_{x^2+y^2}^{2-x^2-y^2} (x^2+y^2)^{3/2}\, dz\, dy\, dx$ (Ans: $\frac{8\pi}{35}$)
    (b) Use spherical coordinates to evaluate $\displaystyle\int_{0}^{1} \int_{0}^{\sqrt{1-x^2}} \int_{0}^{\sqrt{1-x^2-y^2}} (x^2+y^2+z^2)^2\, dz\, dy\, dx$ (Ans: $\frac{\pi}{14}$)

47. Sketch the region of integration of the following integrals.
    (a) $\displaystyle\int_0^{2\pi} \int_0^2 \int_{2r^2}^8 r\, dz\, dr\, d\theta$ (b) $\displaystyle\int_0^{2\pi} \int_0^2 \int_0^8 r\, dz\, dr\, d\theta$ (c) $\displaystyle\int_0^{2\pi} \int_0^2 \int_0^8 r\, dr\, dz\, d\theta$
    (d) $\displaystyle\int_0^{2\pi} \int_0^4 \int_{2r}^8 r\, dz\, dr\, d\theta$ (e) $\displaystyle\int_0^{2\pi} \int_0^1 \int_{r^2}^r r\, dz\, dr\, d\theta$

48. Evaluate the line integral $\int_C x^3 z\, ds$ where C is given by $x = 2\sin t$, $y = t$, $z = 2\cos t$, $0 \leq t \leq \pi/2$. (Ans: $4\sqrt{5}$)

49. Evaluate the line integral $\int_C y\, dx + z\, dy + x\, dz$ where C consists of the line segments from $(0,0,0)$ to $(1,1,2)$ and from $(1,1,2)$ to $(3,1,4)$. (Ans: $\frac{17}{2}$)

50. Evaluate the line integral $\int_C x\sqrt{y}\, dx$ where C consists of the shortest arc of the circle $x^2 + y^2 = 1$ from (-1,0) to (0,1). (Ans: $-\frac{2}{5}$)

51. Find the work done by the force field $\mathbf{F}(x,y,z) = x \sin y\, \mathbf{i} + y\, \mathbf{j}$ on a particle that moves along the parabola $y = x^2$ from (-1,1) to (2,4). (Ans: $\frac{1}{2}(15 + \cos 1 - \cos 4)$)

52. A constant force $\mathbf{F} = 3\,\mathbf{i} + 5\,\mathbf{j} + 10\,\mathbf{k}$ moves an object along the line segment from (1,0,2) to (5,3,8). Find the work done if the distance is measured in meters and the force in newtons. (Ans: 87 joules)

53. Evaluate the line integral $\int_C (3x^2yz - 3y)\, dx + (x^3z - 3x)\, dy + (x^3y + 2z)\, dz$ where C is the curve shown in the Figure 4. (Ans: -4)

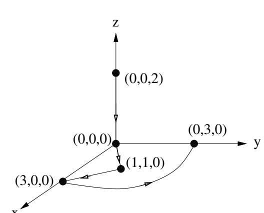


FIG 4

54. Let $\mathbf{F}(x,y) = (2x^3 + 2xy^2 - 2y)\mathbf{i} + (2y^3 + 2x^2y - 2x)\mathbf{j}$. Evaluate the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$ where C is the curve shown in Figure 5.

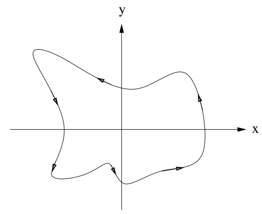


FIG 5

V-VI. VECTOR FIELDS AND GREENS THEOREM

55. Sketch the vector fields.

(a) $\mathbf{F}(x,y) = \langle -1, 2\rangle$ (b) $\mathbf{F}(x,y,z) = \mathbf{i} + \mathbf{k}$ (c) $\mathbf{F}(x,y,z) = z\,\mathbf{j}$

(d) $\mathbf{F}(x,y) = x\,\mathbf{i} + y\,\mathbf{j}$ (e) $\mathbf{F}(x,y) = \dfrac{\langle x, y\rangle}{\sqrt{x^2+y^2}}$ (f) $\mathbf{F}(x,y) = \langle y, -x\rangle$

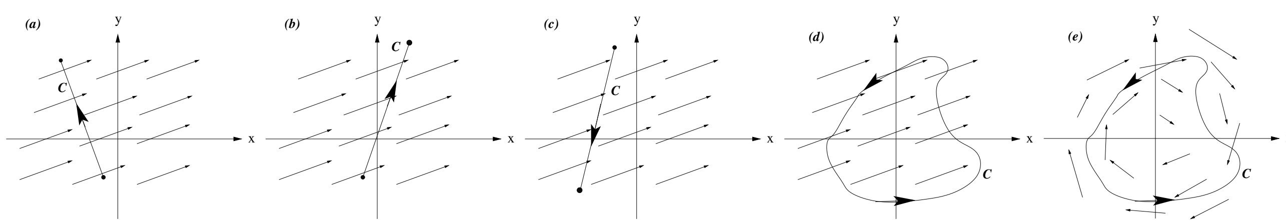
(g) $\mathbf{F}(x,y) = \dfrac{y\,\mathbf{i} - x\,\mathbf{j}}{\sqrt{x^2+y^2}}$

56. *Find and sketch* ∇f. §16.1: 29, 30

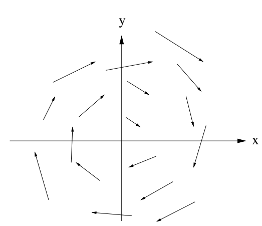
57. *Find a potential function.* Chapter 16 Review, Exercises: 11,12.

58. Show that the vector field $\mathbf{F} = \langle 3x^2yz - 3y, x^3z - 3x, x^3y + 2z\rangle$ is conservative. Find the potential function f.

59. The following figures show a vector field $\mathbf{F}$ and an oriented curve C. From the following figures, estimate whether $\int_C \mathbf{F}\cdot d\mathbf{r}$ is positive, negative or zero. Explain your answer.



60. Is the field in the figure below conservative? Why or why not?



61. Consider the vector field $\mathbf{F}(x,y) = \langle x, -y\rangle$.

(a) Show that $\mathbf{F}$ is conservative and find a potential function f.

(b) Sketch some level curves of f along with the vector field $\nabla f = \mathbf{F}$.

62. Use Green's Theorem to evaluate

$$\int_C \sqrt{1+x^3}\,dx + 2xy\,dy$$

where C is the triangle with vertices $(0,0)$, $(1,0)$, and $(1,3)$ (positively oriented).
(Ans: 3)

63. Use Green's Theorem to evaluate $\int_C x^2y\,dx - xy^2\,dy$, where C is the circle $x^2+y^2=4$ with counter-clockwise orientation.
(Ans: -8π)

64. Find curl $\mathbf{F}$ and div $\mathbf{F}$ if $\mathbf{F}(x,y,z) = x^2z\,\mathbf{i} + 2x\sin y\,\mathbf{j} + 2z\cos y\,\mathbf{k}$.

65. Consider the vector field $\mathbf{u}(x,y,z) = \langle 1, y^2, 0\rangle$

    (a) Sketch a representative set of vectors in the xy-plane.
    (b) Find curl $\mathbf{u}$ and div $\mathbf{u}$.
    (c) Where is the field compressing? Where is it expanding?
    (d) Draw a closed curve C, oriented counterclockwise, on top of your vector field. What is $\int_C \mathbf{u}\cdot\mathbf{T}\,ds$? Why?

66. (a) Show that if $\mathbf{F}$ is a gradient field (i.e., conservative), then curl $\mathbf{F} = \mathbf{0}$.
    (b) Is $\mathbf{F}(x,y,z) = x^2z\,\mathbf{i} + 2x\sin y\,\mathbf{j} + 2z\cos y\,\mathbf{k}$ a gradient field?

67. (a) Show that if $\mathbf{F}$ is a curl (i.e., $\mathbf{F}$ = curl $\mathbf{G}$ for some $\mathbf{G}$), then div $\mathbf{F} = 0$.
    (b) Is $\mathbf{F}(x,y,z) = x^2z\,\mathbf{i} + 2x\sin y\,\mathbf{j} + 2z\cos y\,\mathbf{k}$ a curl?

68. *Curl and div.* Chapter 16 Review, Exercises: 18, 19.