Math 2531 - Exam 3 Review - Fall 2024

Exam 3 covers §15.1-15.4, 15.6-15.8, 16.1-16.4; HW 9-13 (unless told otherwise by your instructor)

Topics:

1. **Double and Triple Integrals**:
   - Approximate double integrals using Riemann sums.
   - Set up and evaluate integrals in Cartesian, polar/cylindrical or spherical coordinates.
   - Sketch the region of integration.
   - Change the order of integration, if necessary, to simplify integration.
   - Change from Cartesion to cylindrical or spherical, if necessary to simplify integration.
   - Applications: computing area, volume, mass, center of mass.
2. **Vector Fields**: Sketch simple vector fields in $\mathbb{R}^2$ and $\mathbb{R}^3$. Compute the gradient field ∇f given $f(x,y)$ or $f(x,y,z)$. Determine whether $\mathbf{F}(x,y)$ is conservative and if so, find a potential function $f(x,y)$ such that $\mathbf{F} = \nabla f$.
3. **Line Integrals of Scalar Functions**: $\int_C f(x,y,z)ds$. Parametrize the curve C and evaluate (be able to parametrize circles, ellipses, curves $y = f(x)$, lines)
4. **Line Integrals of Vector Fields**: $\int_C \mathbf{F}\cdot\mathbf{T}\,ds = \int_C \mathbf{F}\cdot d\mathbf{r} = \int_C \mathbf{F}\cdot\mathbf{r}'(t)\,dt = \int_C P\,dx + Q\,dy + R\,dz$
   (a) Evaluate when $\mathbf{F}$ is not conservative (need parametrization of C)
   (b) Evaluate when $\mathbf{F}$ is conservative, i.e. $\mathbf{F} = \nabla f$ (use Fundamental Theorem for Line Integrals).
   (c) Application: Compute the work done by a force field $\mathbf{F}$ in moving a particle along a path C.
5. **Green's Theorem**: For positively oriented, piecewise-smooth, simple closed curve C in the xy-plane enclosing region D, if P and Q have continuous partial derivatives on an open region containing D, then
$$\int_C P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

Sample Problems:

1. Chapter 15 Review Exercises: 2
2. Chapter 15 Review Exercises: 9, 10, 25, 34, 42b
3. Let $R = [0,2]\times[0,4]$. The integral $\iint\limits_R (9 - x^2)\,dA$ represents the volume of a solid. Sketch that solid and compute the volume.
4. Evaluate the integral $\int_0^1 \int_y^1 \cos(x^2)\,dx\,dy$
5. Find the volume bounded by the surfaces $y = x^2 + z^2$ and $y = 3$.
6. Find the volume of the solid above $z = \sqrt{3x^2 + 3y^2}$ and below $x^2 + y^2 + z^2 = 4$.
7. Let E be the tetrahedron bounded by $x = 0$, $y = 0$, $z = 0$ and $2x + y + z = 2$. Set up the integral $\iiint\limits_E y\,dV$ using 3 different orders: $dV = dz\,dy\,dx$, $dV = dx\,dy\,dz$, $dV = dy\,dx\,dz$.
8. Find the coordinates of the center of mass of the solid hemisphere of radius a: $x^2 + y^2 + z^2 \le a^2$, $z \ge 0$, if the density is constant.

9. Find the volume of the region in the first octant bounded by $y^2 + z^2 = 9$, $x = 0$, $y = 3x$, $z = 0$.

10. Evaluate the integral $\iiint_E z\,dV$, where E is the region in the first octant bounded by $y = 0$, $z = 0$, $x = 0$, $x + y = 2$, $y^2 + z^2 = 1$.

11. Evaluate the integral $\iiint_E yz\,dV$ where E lies above the plane $z = 0$, below the plane $z = y$, and inside the cylinder $x^2 + y^2 = 4$.

12. (a) Set up the integral $\iiint_E (x^2+y^2+z^2)\,dV$ where E is the region bounded below by the cone $\phi = \pi/6$ and above by the sphere $\rho = 2$ using: (i) Cartesian coordinates, (ii) cylindrical coordinates, and (iii) spherical coordinates.
    (b) Evaluate the integral.

13. Consider the integral $\int_0^{\pi/2} \int_0^1 \int_0^{\sqrt{3}} r(1 - r^2)\,dz\,dr\,d\theta$.
    (a) Sketch the region of integration.
    (b) Write the integral in Cartesian coordinates and spherical coordinates.
    (c) Evaluate the integral (use whichever coordinate system you like).

14. Sketch the following vector fields.
    (a) $\mathbf{F}(x, y) = \langle -1, 2\rangle$
    (b) $\mathbf{F}(x, y) = \frac{\langle x,y\rangle}{x^2+y^2}$
    (c) $\mathbf{F}(x, y) = \langle y, -x\rangle$

15. Evaluate the line integral $\int_C x^3 z\,ds$ where C is given by $x = 2\sin t$, $y = t$, $z = 2\cos t$, $0 \le t \le \pi/2$.
    Ans: $4\sqrt{5}$

16. Evaluate the line integral $\int_C y\,dx + z\,dy + x\,dz$ where C consists of the line segments from $(0, 0, 0)$ to $(1, 1, 2)$ and from $(1, 1, 2)$ to $(3, 1, 4)$.
    Ans: $\frac{17}{2}$

17. Evaluate the line integral $\int_C x\sqrt{y}\,dx$ where C consists of the shortest arc of the circle $x^2 + y^2 = 1$ from (-1,0) to (0,1).
    Ans: $-\frac{2}{5}$

18. Find the work done by the force field $\mathbf{F}(x, y, z) = x \sin y\,\mathbf{i} + y\,\mathbf{j}$ on a particle that moves along the parabola $y = x^2$ from (-1,1) to (2,4).
    Ans: $\frac{1}{2}(15 + \cos 1 - \cos 4)$

19. A constant force $\mathbf{F} = 3\mathbf{i} + 5\mathbf{j} + 10\mathbf{k}$ moves an object along the line segment from (1,0,2) to (5,3,8). Find the work done if the distance is measured in meters and the force in newtons.
    Ans: 87 joules

20. Determine whether $\mathbf{F}(x, y) = (1 + xy)e^{xy}\,\mathbf{i} + (e^y + x^2 e^{xy})\,\mathbf{j}$ is conservative. If so, find a potential f such that $\mathbf{F} = \nabla f$.

21. Find a potential function f for the conservative vector field $\mathbf{F} = \langle 3x^2yz - 3y, x^3z - 3x, x^3y + 2z\rangle$.

22. Evaluate the line integral $\int_C (3x^2yz - 3y)\, dx + (x^3z - 3x)\, dy + (x^3y + 2z)\, dz$ where C is the curve shown in the Figure 4.
Ans: -4

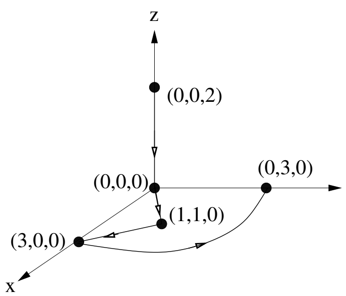


FIG 4

23. Let $\mathbf{F}(x, y) = (2x^3 + 2xy^2 - 2y)\mathbf{i} + (2y^3 + 2x^2y - 2x)\mathbf{j}$. Evaluate the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$ where C is the curve shown in Figure 5.

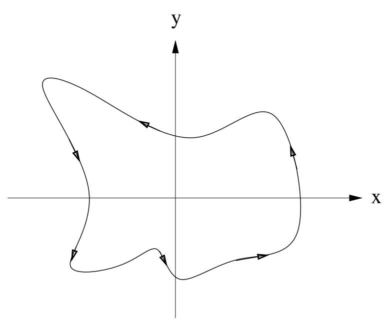
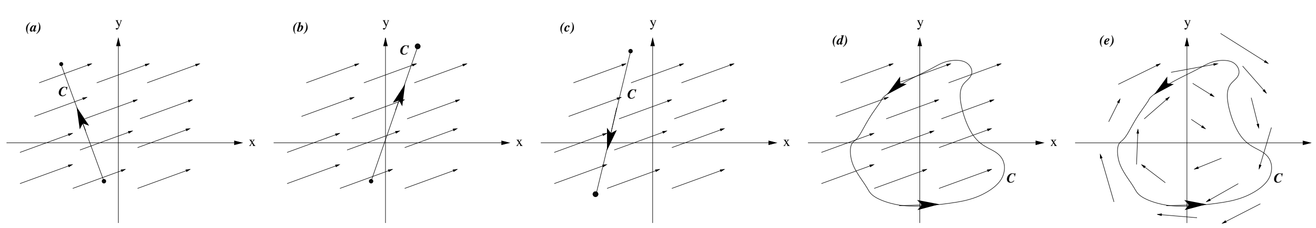
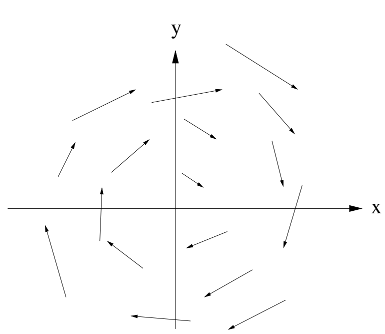


FIG 5

24. The following figures show a vector field $\mathbf{F}$ and an oriented curve C. Estimate whether $\int_C \mathbf{F} \cdot d\mathbf{r}$ is positive, negative or zero.



25. Is the field in the figure below conservative? Why or why not?



26. Use Green's Theorem to evaluate

$$\int_C \sqrt{1+x^3}\,dx + 2xy\,dy$$

where C is the triangle with vertices $(0,0)$, $(1,0)$, and $(1,3)$ (positively oriented).
Ans: 3

27. Use Green's Theorem to evaluate $\int_C x^2y\,dx - xy^2\,dy$, where C is the circle $x^2+y^2=4$ with counterclockwise orientation.
Ans: -8π