Last time delta method
Today: review, S 6, maybe 6.1

Another example for delta method.

Let \( X \sim \text{exp} \) with mean \( \beta \).

What is \( E[\sqrt{X}] \) and \( \text{Var}(\sqrt{X}) \)?

First-order delta method approximation gives
\[
E[\sqrt{X}] \approx \sqrt{E(X)} = \sqrt{\beta}
\]

\[
E[\sqrt{X}] = \int_0^\infty g(x)f(x)\,dx
\]

\[
= \frac{1}{\beta} \int_0^\infty \sqrt{x} e^{-x/\beta} \, dx
\]

\[
= \frac{1}{\beta} \int_0^\infty x^{1/2} e^{-x/\beta} \, dx
\]

Looks like gamma \((\frac{3}{2}, \beta)\)

Recall for a gamma, density
\[
\frac{1}{\Gamma(\frac{3}{2}) \beta^{3/2}} x^{3/2-1} e^{-x/\beta} \, dx
\]

\[\Rightarrow \text{gamma}(1.5) = \frac{\Gamma(1.5) \beta^{3/2}}{\beta^{1.5}} = \sqrt{\beta} \cdot \Gamma(1.5) \approx 0.886 \sqrt{\beta}\]
\[ \text{Var}(\sqrt{x}) \approx \left[ g'(\mu) \right]^2 \text{Var}(x) \frac{\beta^2}{\mu^2} \]

\[ g(y) = \sqrt{y} \quad g'(y) = \frac{1}{2\sqrt{y}} \]

\[ g'(y) \Big|_{y=\mu} = \frac{1}{2\sqrt{\mu}} = \frac{1}{2\sqrt{\beta}} \]

So \[ \text{Var}(\sqrt{x}) \approx \left[ g'(\mu) \right]^2 \text{Var}(x) \]

\[ = \left( \frac{1}{2\sqrt{\beta}} \right)^2 \beta^2 \]

\[ = \frac{1}{4} \beta \]

Could check using

\[ \text{Var}(\sqrt{x}) = E[(\sqrt{x})^2] - (E[\sqrt{x}])^2 \]

\[ = E(x) - (E(\sqrt{x}))^2 \]

\[ = \beta - \left( \Gamma\left(\frac{3}{2}\right) \beta \right)^2 \]

\[ = \beta - \frac{\beta}{\Gamma\left(\frac{3}{2}\right)^2} \]

\[ = \beta \left( 1 - \frac{1}{\Gamma\left(\frac{3}{2}\right)^2} \right) \]
Generally, how to compute or estimate $E[g(X)]$.

Strategies

1. $E[g(X)] = \int g(x)f_X(x)\,dx$

2. Delta method: $E[g(X)] \approx g(\mu) \left(1 + \frac{g''(\mu)}{2} \text{Var}(X)\right)$

   where
   
   $g(x) = g(\mu) + g'(\mu)(x-\mu) + \frac{g''(\mu)}{2} (x-\mu)^2 + R_x^2$

3. Let $Y = g(X)$.

   Derive the density for $Y$

   Use $E[Y] = \int f_Y(y)\cdot y\,dy$

4. Use simulation.

   In R
   
   ```R
   X <- rexp(10000)
   mean(1/sqrt(X))
   ```
Generally, how to compute or estimate $E[g(X)]$.

Strategies:

1. $E[g(X)] = \int g(x) f_X(x) \, dx$

2. Delta method: $E[g(X)] \approx g(\mu) \left( \frac{\text{Var}(X)}{\mu^2} \right)$

   or second-order $\approx g(\mu) + \frac{g''(\mu)}{2} \text{Var}(X)$

   

   $g(x) = g(\mu) + g''(\mu)(x-\mu) + \frac{g''''(\mu)}{2} (x-\mu)^2$

   

   $E[g(X)] = g(\mu) + g''(\mu) \text{Var}(X)$

3. Let $Y = g(X)$.

   Derive the density for $Y$

   Use $E[Y] = \int f_Y(y) \cdot y \, dy$

4. Use simulation.

   In R

   ```
   > X <- rexp(10000)
   > mean(sqrt(X))
   ```
Simulating from a nonstandard density

Example. Suppose $X \sim \text{Beta}(2, 6)$.

Suppose $Y$ has density

$$f_Y(y) = \frac{1}{y^2} \mathbf{1}(\frac{1}{2} \leq y \leq 1)$$

To simulate $Y$ which has density $f_Y(y)$, generate $U \sim \text{U}(0, 1)$

Then let $Y = F_Y^{-1}(U)$

Also use cdf to check why this works.

$$F_Y(F_Y^{-1}(U)) = u$$

$$= P[F_Y(F_Y^{-1}(U)) \leq F_Y(U)]$$

$$= P[U \leq F_Y(U)]$$

For $u \in (0, 1)$

$$P[U \leq u] = u$$

$$= F_Y(u)$$

$\frac{1}{6}$
Let $X \sim U(1,2)$
$y = \frac{1}{X}$.

Then
$$f_Y(y) = \frac{1}{y^2} I(\frac{1}{2} \leq y \leq 1)$$
$$F_Y(y) = \int_{\frac{1}{2}}^{y} \frac{1}{t^2} dt$$
$$= -t^{-1} \bigg|_{\frac{1}{2}}^{y}$$
$$= -\frac{1}{y} - -\frac{1}{\frac{1}{2}}$$
$$= 2 - \frac{1}{y} I(\frac{1}{2} \leq y \leq 1)$$

For the inverse cdf
$$x = 2 - \frac{1}{y} \text{ solve for } y$$
$$\frac{1}{y} = 2 - x \Rightarrow y = \frac{1}{2 - x}$$
$$F_y^{-1}(y) = \frac{1}{2 - y} \quad y = F_y^{-1}(U) = \frac{1}{2 - U}$$

to generate $Y$ use
$$U \sim \text{unif}(0,1)$$
$$y \sim \frac{1}{2 - U}$$
Suppose $X \sim \text{exp}$. What is density of $Y = \frac{1}{1 + X}$?

$x \sim \text{exp}(1000)$

$\text{hist}(x/(1+x))$

$F_x^{-1}(U)$
Ch. 6 Principles of data reduction.

If $X_1, \ldots, X_n$ is a random sample, then $T(X)$, a statistic of $X$, is often a summary of the data.

Examples: 5-number summary ($\min, 25\%, 50\%, 75\%, \max$) box plot

- $\bar{X}$
- median
- mode
- $X_{(n)}/X_{(1)}$

Usually, we're interested in statistics that preserve the useful information in the data for making inferences about parameters.

Example. Suppose $X_1, X_2 \sim \text{Pois}(\lambda)$. We want to figure out/estimate $\lambda$. If I just tell $\bar{X}$, is that enough?

Suppose $\bar{X} = 2$. Data could have been $(0,4), (2,2), (1,3), (3,1), (4,0)$
A statistic partitions the set of all possible data sets.

\[ A_0 = \{ x : T(x) = 0 \} \]

For Poisson example, \( T(x) = \frac{x_1 + x_2}{2} \)

\[ A_1 = \{ (0, 4), (1, 3), (2, 2), (3, 1), (4, 0) \} \]

\[ A_2 = \{ (0, 2), (1, 1), (2, 0) \} \]

\[ A_3 = \{ (0, 0) \} \]

\[ A_{1/2} = \{ (1, 0) \} \]

Suppose \( T(x) = x_1 \).

**Sufficiency Principle.** If \( T(x) \) is a sufficient statistic (SS) (sufficient for \( \theta \)), then any inference about \( \theta \) should depend on \( x \) only through \( T(x) \).

If \( x \) and \( y \) are two data sets where \( T(x) = T(y) \), then any inference about \( \theta \) should be the same for \( x \) and \( y \).