Ancillary statistics

A statistic \( S(x) \) is called ancillary if it provides no information about \( \theta \).

Example: \( X_1, \ldots, X_n \overset{iid}{\sim} \mathcal{N}(\mu, 1) \)

Let \( Z_i = \frac{X_i - \overline{X}}{\frac{1}{\sqrt{n}}} \)

No information about \( \mu \) from \( Z_i \).

What is the distribution of \( Z_i \)?

Since \( Z_i \) is normal, its distribution is specified by its mean and variance

\[
E[Z_i] = E[X_i - \overline{X}]
= E[X_i] - E[\overline{X}]
= \mu - \mu = 0
\]

\[
\text{Var}(Z_i) = \text{Var}(X_i - \overline{X}) = \text{var}(X_i) + \text{var}(\overline{X}) - \text{cov}(X_i, \overline{X})
= \text{var}(X_i - \frac{1}{n} \sum_{i=1}^{n} X_i)
= \text{var}(X_i - \frac{1}{n} X_1 - \frac{\sum_{i=2}^{n} X_i}{n})
= \text{var}((n-1)X_i - \sum_{i=2}^{n} X_i)
= (\frac{n-1}{n})^{\frac{2}{n}} \text{var}(X_i) + \text{var}(\sum_{i=2}^{n} X_i)
= (\frac{n-1}{n})^{\frac{2}{n}} + (n-1)\frac{1}{2} \quad \text{doesn't depend on } \mu.
\]
\[ y_i \sim U(0, 1) \]
\[ x_i = y_i + \theta \]

Another example \( x_1, \ldots, x_n \sim U(\theta, \theta + 1) \)

\( x_{(n)} - x_{(1)} \) is ancillary for \( \theta \).

Generally for location families,
\( x_{(n)} - x_{(1)} \) is ancillary for \( \theta \).

For scale families
\[
\frac{x_1}{x_n} \ldots \frac{x_{n-1}}{x_{(n)}}
\]

or
\[
\frac{x_1}{x_{(n)}} \ldots \frac{x_n}{x_{(n)}}
\]

are ancillary.

For \( U(0, \theta) \) case

let \( x_1, \ldots, x_n \sim U(0, \theta) \)

let \( \delta \) \( x_i = \theta u_i \), \( u_i \sim U(0, 1) \)

\[
\frac{x_i}{x_n} = \frac{\theta u_i}{\theta u_n} = \frac{u_i}{u_n}
\]

Exponentials are also scale families

If \( x \sim \exp(1) \)
\( \delta x \sim \exp \delta \text{ w/ mean } \theta \).
Intuitively, ancillary statistics are

"opposite" of sufficient statistics.

You might expect ancillary statistics to be independent of sufficient statistics. Often true, but not always.

The concept of complete statistics helps fill the gap. Ancillary statistics are independent of complete, sufficient statistics.

**Defn. 6.2.2** Let \( f(t; \theta) \) be a family of distributions for \( T(X) \). The family is called complete if

\[
\mathbb{E}[g(T)] = 0 \text{ implies } \mathbb{E}[g(T; \theta)] = 1 \text{ for all } \theta.
\]

**Example** (check for completeness)

Let \( X_1, \ldots, X_n \sim U(0, \theta) \)

\( T(X) = X_{(n)} \). We know \( T(X) \)

is sufficient (previous examples).

We want to check for completeness.

\[
f(t; \theta) = \begin{cases} \frac{n t^{n-1}}{\theta^n} & 0 < t < \theta \\ 0 & \text{otherwise} \end{cases}
\]
we want to show that \( T(x) = X_n \) is a C.S.S.

Suppose \( g \) is a function satisfying

\[
E[g(T)] = 0.
\]

Since \( E[g(T)] \) is constant w.r.t. \( \theta \)

\[
\frac{d}{d\theta} E[g(T)] = \frac{d}{d\theta} 0 = 0.
\]

due to

\[
E[g(T)] = \int g(t) f(t|\theta) dt
\]

so

\[
0 = \frac{d}{d\theta} 0 = \frac{d}{d\theta} E[g(T)] = \frac{d}{d\theta} \int g(t) f(t|\theta) d\theta
\]

leibniz's rule

\[
= \frac{d}{d\theta} \int_0^\theta g(t) t^{n-1} \theta^n d\theta
\]

or

\[
= \frac{d}{d\theta} \left[ \int_0^\theta f(x, \theta) dx \right]^{\theta}
\]

\[
= f(x, \theta) \bigg|_{x=0}^{x=\theta} + \int_0^\theta \frac{d}{d\theta} f(x, \theta) dx
\]

\[
= g(\theta) n \theta^{n-1} \theta^{-n} + \theta \int_0^\theta \frac{d}{d\theta} g(t) t^{n-1} \theta^n d\theta
\]

and

\[
g(\theta) n \theta^{n-1} = 0 \Rightarrow g(\theta) = 0.
\]
Basu Thm. 6.2.24

If \( T(X) \) is a complete s.s. then it is independent of every ancillary statistic.

Example \( \overline{X} \) and \( S^2 \) are for

\[ a, b \text{ i.i.d. normal } \Rightarrow a, b \text{ are independent because} \]

\[ S^2 = \frac{1}{n-1} \sum (X_i - \overline{X})^2 = \frac{1}{n-1} \sum X_i^2 \]

so \( S^2 \) is ancillary for \( \mu \).

\( \overline{X} \) is a c.s.s. \( \Rightarrow \overline{X} \) is ind. of \( S^2 \).

Ex. let \( X_1, \ldots, X_n \) i.i.d. \( \exp \) w. mean \( \theta \).

Then \( \frac{X_n}{\overline{X}} \) is ancillary for \( \theta \)

\[ \frac{X_n}{\overline{X}}; \frac{\sum X_i}{\overline{X}}; \] is the proportion of \( \sum X_i \) contributed by \( X_n \).

\[ \sum_{i=1}^n X_i \] is a c.s.s.

So \( \overline{X} \) is ind. of \( \frac{X_n}{\sum X_i} \).
Thm 6.2.25

If $X_1, \ldots, X_n$ is an iid sample from an exp fam. with $\Theta = (\theta_1, \ldots, \theta_k)$

then $T(X) = \left( \frac{1}{n} \sum_{i=1}^{n} t_1(X_i), \ldots, \frac{1}{n} \sum_{i=1}^{n} t_k(X_i) \right)$

is a c.s.s. as long as the parameter space $\Theta$ contains an open set in $\mathbb{R}^k$.

This thm doesn't apply to curved exponential families such as

$\theta \sim \text{normal}(\theta^0, \sigma^2)$

$\text{gamma}(x, \alpha)$

Thm 6.2.28 If a minimal s.s. exists, then any c.s.s. is a m.s.s.
§ 6.3. Likelihood

If \( f(x|\theta) \) is the joint density or pmf for a sample

\[ L(\theta | x) \]

is the likelihood.

\[ L(\theta | x) = f(x | \theta) \]

think of parameter as fixed, data as variable

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Suppose you have one observation of an exponential w. mean \( \beta \).

\[ f(x | \beta) = \frac{1}{\beta} e^{-x/\beta} \]

Let \( \beta = 1 \)

Suppose \( x = 1 \)

\[ L(\beta | x) = \frac{1}{\beta} e^{-1/\beta} \]

\[ \int_{\beta} e^{-x/\beta} \, d\beta \]