Math & Music

I

Chapter 1

Introduction to rhythm and the rhythmic pyramid

We begin with the idea of the rhythmic pyramid. The idea is that given any duration, we could either double it or split it into two equal halves, like cutting a slice of cake.

Much of the music we listen to in rock, blues, jazz, classical, and other traditions is based on phrases that are four beats long. A *whole note* has this duration of four beats. We also have

- A *half note* has a duration of half of a whole note, or two beats.
- A *quarter note* lasts for one-half of a half-note, or one beat.
- An *eighth note* lasts for half as long as a quarter note
- A *sixteenth note* last for half as long as an eighth note

In principal, there's no end to the number of subdivisions you can imagine — 64th notes, 128th notes, etc. We think of the whole note at the top of the pyramid. But we can work from the bottom to the top as well, by saying that a quarter note lasts twice as long as an eighth note, for example.

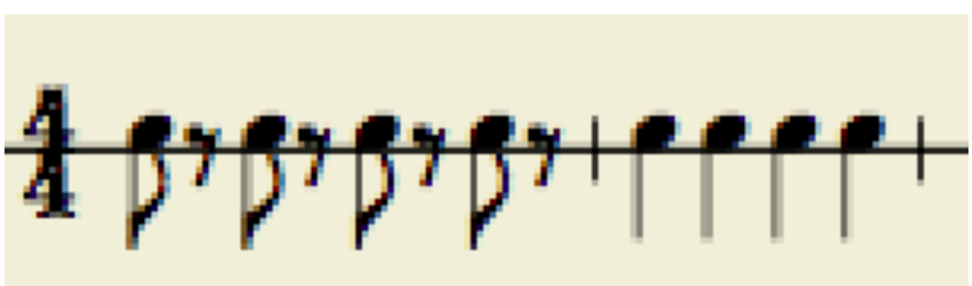
We can also say that a quarter note has the same duration as two eighth notes, an eighth note has the same duration as two sixteenth notes, and so forth.

We can write these relationships this way

$$\begin{aligned}
\text{𝅝} &= \text{𝅗𝅥} + \text{𝅗𝅥} \\
\text{𝅗𝅥} &= \text{𝅘𝅥} + \text{𝅘𝅥} \\
\text{𝅘𝅥} &= \text{𝅘𝅥𝅮} + \text{𝅘𝅥𝅮} \\
\text{𝅘𝅥𝅮} &= \text{𝅘𝅥𝅯} + \text{𝅘𝅥𝅯} \\
\text{𝅘𝅥𝅯} &= \text{𝅘𝅥𝅰} + \text{𝅘𝅥𝅰}
\end{aligned} \tag{1.1}$$

For the equations above, it is important to remember that this is intended to just mean that the durations are the same, not that they sound the same.

For percussion instruments such as drums, a note that is written as a duration such as a quarter note or a half note does not necessarily last that long. This can be true of a guitar as well when the sustain is not very long—the note might fade too fast to be heard for the entire written duration. As a result the rhyths in the following two measures of 4/4 time might sound identical when played on percusion instrument or a guitar with short sustain at a slow tempo.



We can express the same relationships as fractions

$$\begin{aligned}1 &= \frac{1}{2} + \frac{1}{2} \\ \frac{1}{2} &= \frac{1}{4} + \frac{1}{4} \\ \frac{1}{4} &= \frac{1}{8} + \frac{1}{8} \\ \frac{1}{8} &= \frac{1}{16} + \frac{1}{16} \\ \frac{1}{16} &= \frac{1}{32} + \frac{1}{32}\end{aligned}$$

Another important rhythmic concept is that of *dotting* notes. A dotted note is equal to one-and-a-half, or 50%, longer duration than the note without the dot. For example, a dotted quarter note, ♩., is 50% longer than a quarter note, ♩.

We can make this more precise using the rhythmic pyramid. Since a quarter note is equivalent in duration to two eighth notes, a dotted quarter note is equivalent in duration to a quarter note plus an eighth notes, or to three eighth notes

♩. = ♩ + ♪
   = ♪ + ♪ + ♪

Rhythmic durations can be added just as fractions are added in math. For example, the mathematical statement that

$$\begin{aligned}\frac{1}{4} + \frac{1}{8} &= \frac{2}{8} + \frac{1}{8} \\ &= \frac{3}{8}\end{aligned}$$

is similar to the musical statement that

♩ + ♪ = ♩.

Dotted versions of whole notes, half-notes, quarter notes, eighth notes, and so forth can all be used, and they all follow a similar rhythm pyramid as their undotted versions:

𝅝. = 𝅗𝅥. + 𝅗𝅥.

𝅗𝅥. = ♩. + ♩.

♩. = ♪. + ♪.

♪. = 𝅘𝅥𝅯. + 𝅘𝅥𝅯.

𝅘𝅥𝅯. = 𝅘𝅥𝅰. + 𝅘𝅥𝅰.

We can also write

𝅝 = 𝅗𝅥 + 𝅗𝅥

= 𝅗𝅥 + ♩ + ♩

= 𝅗𝅥 + ♩ + ♪ + ♪

= 𝅗𝅥 + ♩ + ♪ + 𝅘𝅥𝅯 + 𝅘𝅥𝅯

= 𝅗𝅥 + ♩ + ♪ + 𝅘𝅥𝅯 + 𝅘𝅥𝅰 + · · ·

If we express this as fractions, we get

$$\begin{aligned}
1 &= \frac{1}{2} + \frac{1}{2} \\
&= \frac{1}{2} + \frac{1}{4} + \frac{1}{4} \\
&= \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{8} \\
&= \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{16} \\
&= \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \cdots
\end{aligned}$$

The sum continues infinitely. In mathematics, this is called an infinite series. In a course in Calculus, you might study this series from a mathematical point of view. From a musical point of view, you are chopping the remaining amount of time in a measure of 4/4 time into two equal pieces over and over and again, so you can imagine making the notes smaller and smaller and still just perfectly filling the measure.

1.0.1 Rhythmic pyramid with triplets

Instead of dividing a unit of time into two equal pieces, we could divide it into three equal pieces. This can occur when there are three beats per musical phrase, such as in a waltz. In this case, we'd say that each measure has 3 beats. Often we divide individual beats into three equal durations, making a triplet.

In triplet-based music, such as much of jazz and blues, however, divisions into three are often then subdivided into two equal, smaller durations. For example, a beat divided into triplets might be divided again by subdividing each triplet into two notes, creating sixteenth-note triplets. This divides a beat into 6 equal durations. If instead, we took a triplet and divided each note of the triplet into three notes, we'd get a subdivision of 1 beat into 9 equal durations. This is possible but less common. A more common case might be a jazz waltz, played with 3 beats per measure, where each beat is divided into triplets, creating 9 notes per measure. Subdividing these triplets into three notes each would create 27 ($= 9 \times 3$) notes per measure.

1.0.2 Rhythmic pyramid with rests

In addition to notes being played, silence can also last the same amount of time as notes being played.

W: 20 Jan 16

Here we explain relationships between whole notes, half notes, quarter notes, eighth notes, 16th notes, and 32nd notes. We also explain dotted notes, but did not explain double dotted notes. Also similar durations for rests. Discussed adding durations as adding fractions including the series

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots = 1$$

I believe that "Clapping Music" by Steve Reich was discussed in this class.

F: 22 Jan 16

Similar to previous lecture but introduced triplets. Songs discussed included "America" from *West Side Story* and *Don't Tread on Me* by Metallica.

Chapter 2

Introduction to permutations

2.1 Circular permuations

Drummers often use the word *permutation* to describe certain types of variations. For example, if you play a paradiddle, meaning a sticking pattern of

$$RLRRLRLL$$

where R means playing with the right hand, and L means playing with the left, a variation on this exercise is the *invertedparadiddle*

$$RLLRLRRL$$

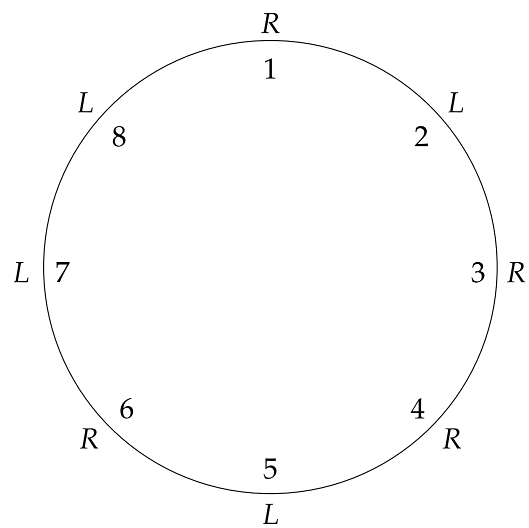
The inverted paradiddle is a permutation of the standard paradiddle. What does this mean?

The two patterns have a lot in common. Both involve playing four Rs and four Ls in a certain sequence, and in both cases, no hand plays more than two notes in a row. In mathematical lingo, a permuation of a sequence of objects (such as letters) would be any rearrangement of the objects into a new sequence. By this meaning of permuation,

$$LLLLRRRR$$

would count as a permutation of the paradiddle.

However, drummers often use *permutation* to mean something more specific, which in mathematical jargon is called a *circular permutation*. For this type of permuation, we imagine the sequence wrapped around a circle. We can start the sequence somewhere on the circle. Starting at a new place on the circle creates a circular permuation of the original sequence.



If we follow the circle at different starting positions, we get different paradiddles:

Starting position	paradiddle
1	*RLRR LRLL* (standard)
3	*RRLR LLRL* (reversed)
4	*RLRL LRLR* (delayed)
6	*RLLR LRRL* (inverted)

Starting at other positions results in left-handed paradiddles.. These paradiddle variations are all circular permutations of each other which we see by looking at the circle. We can also see that there are no other circular permutations of these paradiddles. For example, the pattern

$$RRLL\ RLRL$$

is not a circular permutation of the paradiddle because there is no way to encounter two *R*s followed by two *L*s anywhere on the circle.

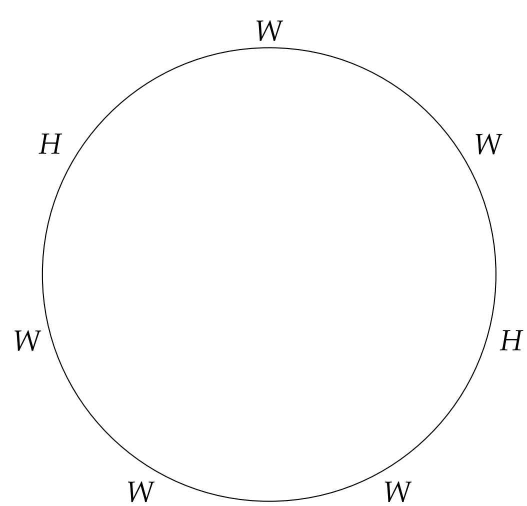
Another approach to thinking of paradiddles as circular permuations of each other is to imagine playing a paradiddle in a loop over and over again. This is similar to the idea of looping around the circle repeatedly. Then you can see paradiddle variations within the standard paradiddle. In the following table, the standard paradiddle is played twice, and variations are put in bold.

sticking	paradiddle variation
RL**RR** **LRLL** **RL**RR LRLL	reversed
RLR**R** **LRLL** **RLR**R LRLL	delayed
RLRR L**RLL** **RLRR** **L**RLL	inverted

This same concept can be used to illustrate how scales and modes are related to each other. If we consider a major scale, such as C major, the space between each note is either a whole step (equivalent to two frets on a guitar or two adjacent keys on a piano, white or black), or a half-step (equivalent to one step on a guitar or one key on a piano). The pattern is

$$WWHWWWH$$

where W means a whole step, and H means a half-step.



Similar to the idea of the paradiddle, we can imagine starting at any of the positions on the circle and moving clockwise one full rotation. Different starting points give us different *modes*. This is similar to the idea of playing all the white notes in an octave of a piano but starting on a different piano key. The following table shows the different modes that result from different starting positions:

Starting position	pattern	mode	styles
1	$WWHWWWH$	Ionian (major)	classical, punk
2	$WHWWWHW$	Dorian	Irish, rock
3	$HWWWHWW$	Phrygian	flamenco, metal
4	$WWWHWWH$	Lydian	
5	$WWHWWHW$	Mixolydian	Irish, rock, punk
6	$WHWWHWW$	Aelian (minor)	classical, metal
7	$HWWHWWW$	Locrian	metal

Different modes are more common in different genres. A rough generalization is that happier-sounding music will tend to have patterns with Hs near the end of the sequence, and sadder or darker music (especially metal) will tend to prefer modes with Hs near the beginning of the sequence. Suppose we score each mode by the sum of the positions of where H occurs. I'll call this the "Happiness Statistic". For example, for major, H occurs in positions 3 and 7, so we'll score it as 10. Mixolydian gets a score of $3 + 6 = 9$. Then we get the following ordering of the modes, which differs from the order of finding them on the circle:

You might notice that the genres of music seem a bit less jumbled up using this organization of the modes as well. This procedure suggests certain creative possibilities. For example, if H occurred in positions 4 and 6, then you'd get a score of 10, which would be the same as Ionian or major. Would this sound similar? Would it be as easy to write energetic-sounding punk riffs in such a scale as in the major scale? Does this correspond to some other scale we haven't encountered yet?

Table 2.1: Modes arranged by the Happiness Statistic

mode	Happiness Statistic	styles
Lydian	$4+7=11$	
Ionian	$3+7=10$	classical, punk
Mixolydian	$3+6=9$	Irish, rock, punk
Dorian	$2+6=8$	Irish, rock
Aeolian	$2+5=7$	classical, metal
Phrygian	$1+5=6$	flamenco, metal
Locrian	$1+4=5$	metal

2.2 Permutations not on a circle

In the concert movie *Stop Making Sense* by the band Talking Heads, first one member of the band, David Byrne, performs a song by himself, singing and playing guitar. Then the bass player, Tina Weymouth, comes out and they play a song with just two performers. For the third song, there are three performers, and for the fourth song there are four performers. Assuming that the band members could have come out in any order, how many ways could the band members have come out one at a time? We'll answer this question in a few paragraphs.

As another application, when a band records an album, they have to decide on the sequence of tracks that will appear on the CD or track listing. If an album has 10 tracks, how many sequences are possible?

There are so many possibilities that it isn't feasible to write them all down. Instead, let's consider a smaller example first. Suppose there only three tracks, call *Air*, *Bat*, and *Cat*. To decide on an order of the tracks, we can list all sequences:

1. Air, Bat, Cat

2. Air, Cat, Bat

3. Bat, Air, Cat

4. Bat, Cat, Air

5. Cat, Air, Bat

6. Cat, Bat, Air

There are six possibilities. But how could we think about this problem more systematically to deal with larger examples?

One way of thinking about it is that there were three choices for the first song. Then once the first song was chosen, there were two choices for the second song, *for each of the three choices of the first song*. This leads to $3 \times 2 = 6$ choices for the first two songs. Once the first two songs are chosen, the third is forced to be last, but we could think of this as $3 \times 2 \times 1 = 6$.

For the Talking Heads concert, there are four band members in the main band (there are also other hired musicians). From the main four members, there were four choices for the first member to appear, three for the second member, two for third, and one choice remaining for the last member. So the total number of sequences possible was

$$4 \times 3 \times 2 \times 1 = 24.$$

For the example of the album with 10 tracks, the number of possible track listings is

$$10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 3,628,800$$

These types of counting problems arise often enough that they are given a special notation, call *factorials*. We read $n!$ as "n factorial. We can think of $n!$ in several ways, depending on what is most convenient:

$$\begin{aligned}
n! &= 1 \times 2 \times \cdots \times n \\
n! &= n \times (n-1) \times (n-2) \times \cdots 2 \times 1 \\
n! &= n \times (n-1)!
\end{aligned}$$

Here are some factorials for small numbers

$$\begin{aligned}
0! &= 1\\
1! &= 1\\
2! &= 2\\
3! &= 6\\
4! &= 24\\
5! &= 120\\
6! &= 720\\
7! &= 5,040\\
8! &= 40,320\\
9! &= 362,880\\
10! &= 3,628,800\\
11! &= 39,916,800\\
12! &= 479,001,600\\
20! &= 2.4 \times 10^{18}\\
30! &= 2.7 \times 10^{32}\\
40! &= 8.2 \times 10^{47}\\
50! &= 3.0 \times 10^{64}\\
60! &= 8.3 \times 10^{81}
\end{aligned}$$

The number 60! is similar to the number of atoms in the universe. If you have a band in your iPOD or iPHONE or other device with 60 songs, and you listen to all of them on shuffle play twice in a row, the chance that you get them all in the same order twice in a row is astronomically small. Winning the Powerball lottery with one ticket would be more likely than using shuffle play on a single album with 12 songs and getting the songs in the same order as the album.

One theme for this course is that possibilities are vast. The number of possible ways of creating music is much larger than what can actually be done. This means that there is much room for creativity.

As another application of permutations, some 20th-century composers such as Arnold Schoenberg had the idea that instead of using traditional musical scales, one could rearrange the 12 notes in an octave in a particular order, called a *tone row*, and play the notes in that order. It is up to the composer to choose the order of the notes, but they are often chosen so as not to emphasize any particular note and to not sound as if the music is written in a particular key. This is a crucial goal for *atonal* music. However, tone-rows could conceivably be chosen with other musical goals in mind. The tone-row concept has occasionally been applied by rock musicians as well, particularly by Ron Jarzombek (Blotted Science).

A natural question to ask is: How many tone rows are possible? Since there are 12 notes possible for the first note, 11 for the second, 10 for the third, and so on, the answer is 12!, nearly half of a billion. Arguably, the first note chosen matters little compared to the intervals between notes. That is, a tone-row created by transposing each note of another tone-row by the same amount (say, one fret on the guitar) will sound very similar. Ignoring the first note, there are 11!, or nearly 40 million tone-rows possible.

We will encounter the idea of tone-rows again later when we talk about ways of transforming music.

2.3 Partial permutations

For a partial permutation, we select some subset of objects from a set, and paying attention to the order in which the objects are selected. For example, if you have time to listen to three tracks from an album that has 10 songs, you could listen to tracks 1, 2, and 3. Or you could listen to tracks 3, 1, and 2, in that order, and that would be a different listening experience. Or you could listen to tracks 3, 10, and 5, and so on.

Partial permutations work the same way as regular permutations, except that instead of multiplying the possibilities starting at n and working all the way down to 1, we stop somewhere between n and 1.

For the example of listening to three tracks out of 10, there are 10 choices available for the first track, 9 remaining for the second track, and 8 for the third track. Thus, the number of ways of listening to 3 songs out of the 10 is

$$10 \times 9 \times 8 = 720$$

assuming that the order of the songs matters.

A special notation for partial permuations is $P(n,k)$ where n is the number of choices available, and k is the number of distinct choices made. A formula is

$$P(n,k) = \frac{n!}{(n-k)!} \tag{2.1}$$

For selecting three items from 10, where the order matters, the formula is

$$P(10,3) = \frac{10!}{7!} = \frac{10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}{7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1} = 10 \times 9 \times 8$$

There is a lot of cancellation in the numerator in the denominator, and instead of using the formula it is easier to think of $P(n,k)$ as having k terms (i.e., k pieces being multiplied). For example,

$$P(n-3) = n \times (n-1) \times (n-2).$$

For $P(n,k)$, the last term being multiplied is $(n-(k-1)) = (n-k+1)$. As a formula, this is

$$\begin{aligned} P(n,k) &= n \times (n-1) \times \cdots \times (n-k+1) \\ &= (n-0) \times (n-1) \times \cdots \times (n-(k-1)). \end{aligned}$$

There are k terms because there are k numbers in the list $0, 1, \ldots, k-1$. However, the form in equation (2.1) is more compact. Which way is easier to use can depend on the application.

M: 25 Jan 16

Types of permutations included circular as applied to drumming, including paradiddles (standard, inverted, delayed, reversed), and the idea that permutations include more than circular permutations. Spent a lot of time counting license plates, with and without distinct letters/numbers.

W: 27 Jan 16

More on permutations, including the formula $P(n,k) = n!/(n-k)!$. Discussed tone-rows and played some Ron Jarzombek doing a tone-row.

F: 29 Jan 16

Scales and modes as permuations, discussing $WWHWWWH$ as the major scale. Also Jazz scales as circular permutations of ascending melodic minor: $WHWWWWH$. Played some scales on a guitar in class.

Chapter 3

Introduction to combinations

3.1 Combinations versus permutations

If you listen to the song "I Would For You" by Jane's Addiction (easily found on Youtube.com), do you notice anything usual about this song by a rock band?

The song starts with just bass and vocals. Guitars never enter the song, even though this is a very guitar-oriented band, and then there are some light synthesizer sounds. Starting a song with just bass and vocals, and not having any drums or guitar, is an unusual *combination* of instruments for a rock band.

Most rock bands have essentially the same instrumentation in every song: guitar, drums, bass, and vocals, maybe keyboards of some kind. For parts of a song, especially an introduction, only a subset of the instruments might play. If there are four instruments in a band, say bass, drums, guitars, and vocals (counting vocals as an instrument), how many ways can only two instruments be playing? How do you list them all? What if there were 6 instruments and three of them were playing? Now how many combinations are there? We'll be answering this type of question in this section.

We use the concept of *combinations* in this setting rather than permutations because the order doesn't matter – bass and vocals are the same two instruments and vocals and bass.

Combinations are similar to partial permutations — we are interested in selecting some subset from a larger set — except that order doesn't matter.

For an everyday example outside of music, if you order a two-topping pizza, ordering a pepperoni and mushroom pizza is the same as ordering a mushroom and pepperoni pizza. The order in which you list the ingredients doesn't matter when you request the pizza. There might be a difference in how the pizza is made – probably one ingredient is put on the pizza before the other, and the order might affect the taste, but this is a decision you leave to the restaurant, and you wouldn't normally request the order in which they place the ingredients.

Combinations and permutations are both useful concepts in both mathematics and music. Which one is more useful depends on the question you have.

M: 1 Feb 16

Introduction to Combinations. Counting pizza toppings. Relationship to permuations,

$$C(n,k) = P(n,k)/k! = \frac{n!}{k!(n-k)!}$$

Jane's Addictions "I would for you" (on the live album) used an example of an unusual combination: bass and vocals (primarily, with slight synth).

W: 3 Feb 16

More on combinations. Binomial theorem, Pascal's triangle, Binomial expansion of $2^n = (1+1)^n$.

$$2^n = (1+1)^n = C(n,0) + C(n,1) + \cdots + C(n,n)$$

F: 5 Feb 16

Ideas from Benny Greb's Language of Drumming on diddling in various places: 16 ways to diddle four sixteenth notes and how to think of this as either 2^4 or $C(4,0) + C(4,1) + C(4,2) + C(4,3) + C(4,4)$. Discussed making variations on "Mary Had a Little Lamb" by diddling some of the notes and played MIDI examples in class. Also played part of "Some of My Favorite Things" by Coltrane.

Chapter 4

Graphs and Music

M: 8 Feb 16

Sick that day.

W: 10 Feb 16

Finished combinations, showed Chris Coleman Gospel Chops Youtube video describing six hand-foot combinations: HHKK, HKHK, HKKH, KKHH, KHKH, KHHK. Described how to count if you distinguish left from right: LRKK, RLKK, etc. as $\frac{4!}{2!1!1!}$ ways. Discussed number of rearrangements of *ALBUQUERQUE* and *MISSISSIPPI*

Discussed how similar musical staff notation is to Cartesian coordinate graphs.

F: 12 Feb 16

There is another sense of a graph that is important in mathematics, which is a set of vertices (also called nodes) and edges. Geometric objects such as triangles, squares, and polygons are often thought of in terms of their vertices and edges rather than in terms of their coordinates in a Cartesian graph.

Graphs in this sense have taken on a new importance in modern life as they are used to represent things like connections between people in

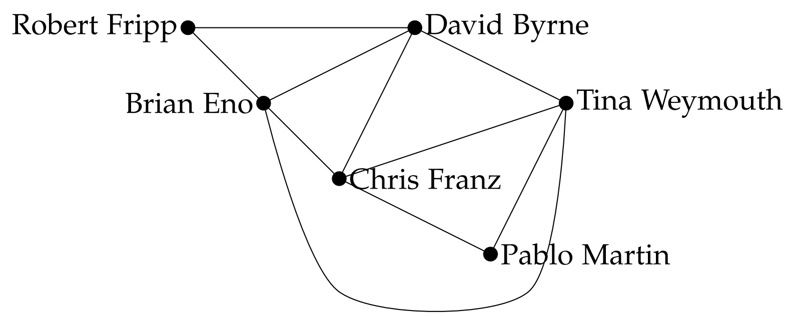


Figure 4.1: A graph depicting relationships between musicians.

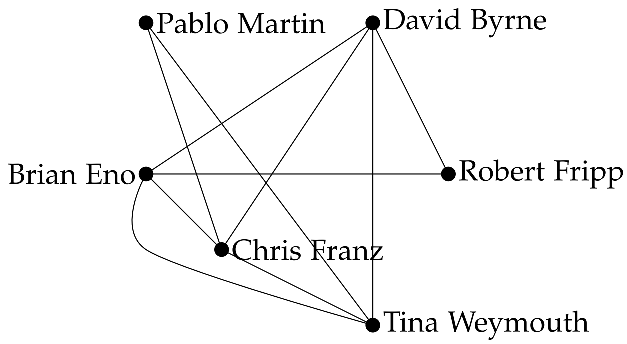
social media accounts, links between websites, and purchasing behavior of customers online (such as when a website says that customers viewing this item also viewed such-and-such a product....). Slightly more traditional uses of such graphs are describe shipping routes such as for a railroad network or airplanes, or to describe telephone connections for calling long distance.

Figure 4.1 is an example depicting a set of musicians. Each node represents a musician, and an edge is drawn if the two musicians have been in the same band or played on at least one album together. As an example, Brian Eno has played with King Crimson and Talking Heads, Tina Weymouth has played in Talking Heads and Tom Tom Club, and Pablo Martin has played in Tom Tom Club. In the diagram, nodes are represented black circles, and are labeled by musicians' names.

The game Six Degrees of Kevin Bacon (REF) illustrates this idea as well, where a graph could have actors as nodes, and an edge means that the actors have appeared in the same movie. The idea is that many actors can be connected within 6 edges from themselves to Kevin Bacon.

From the mathematical point of view, for this type of graph, the locations of the nodes don't matter at all. The nodes and edges for David Byrne, Chris Franz, Pablo Martin, and Tina Weymouth happen to form an irregular quadrilateral (a four-sided polygon), but could have been drawn as a square instead, or even with lines crossing. The lengths of the edges also don't usually matter, or whether edges are drawn straight or curved.

Curved edges might be drawn to make it easier for edges to not cross in the drawing. The following graph depicts exactly the same relationships and so is equivalent to the first one, in spite of lines crossing. This second graph is probably harder to read.



Problems that arise for such graphs include determining the minimum distance from one point to another, the minimum cost from one point to another (which might or might not have the minimum distance), determining nodes can be arranged so that lines don't cross, and predicting which nodes will grow more connections to other nodes (when graphs change over time).

Such graphs can be used in music in different ways. For example, vertices could be used to represent modes of a scale, and two vertices (i.e., two modes) could be connected if the two modes differ by at most one note. This is a useful way to visualize how similar to modes are in terms of how they sound. In particular, for this example, we'll consider modes that start on the same note. We'll compare C ionian (major), C dorian, C phyrgian, C lydian, C mixolydian, C aeolian, and C locrian. Are all of these modes connected by this definition of the graph?

It turns out that two of the seven diatonic modes (Ionian, Dorian, Phrygian, Lydian, Mixolydian, Aeolian, and Locrian) starting on the same note differ by one note if they are adjacent to each other in Table 2.1, which ordered modes by their Happiness Statistic.

We can make a graph of the modes considering two modes to be connected if they differ by exactly one note. Here we show the graph two

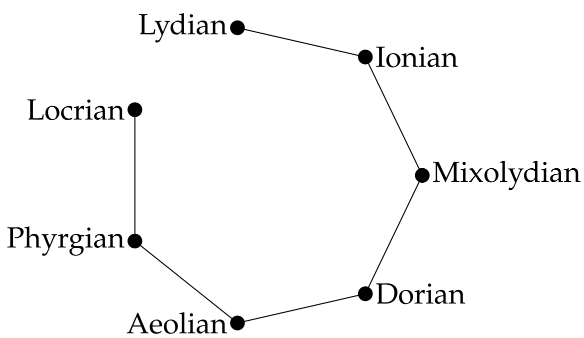


Figure 4.2: Graph of modal relationships. Edges indicate that modes starting on C differ by exactly one note.

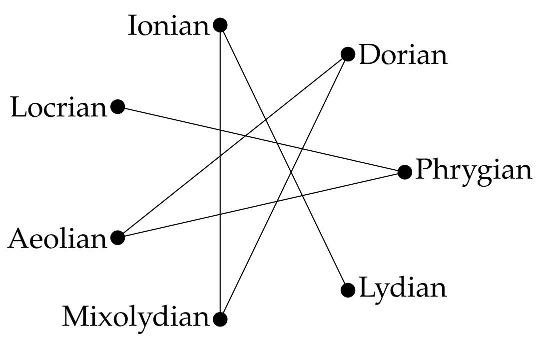


Figure 4.3: Graph of modal relationships. Edges indicate that modes starting on C differ by exactly one note.

ways, ordering the nodes in a circle either based on the Happiness Statistic, or based on the circle showing the distances $WWHWWWH$.

If we draw the modes on a circle in the order in which they are normally presented, based on playing seven white keys in a row first starting on C, then starting on D, etc., then the relationships would be graphed as in Figure 4.3

Both figures look a bit incomplete — the near circular Figure 4.2 needs one edge to complete a loop, and Figure 4.3 needs one edge to complete a seven-pointed star. In both cases apparently missing edge would connect the Lydian and Locrian modes. Is there a connection between these two modes? if you were willing to sharp the first note of C Lydian (so play C Locrian except play C-sharp instead C), you end up with C-sharp Locrian. Similarly, if you flat first note of C Locrian, this is sounds like B Lydian. It's almost as if by flatting the first scale degree, the modified Locrian suddenly becomes happy.

In addition to determining which modes are the most closely related, we can represent which key signatures are most closely related using a graph. Here we describe 12 keys in terms of the numbers of sharps and flats they have:

Table 4.1: Circle of Fifths

Key	sharps/ flats	notes						
A-flat	4	A-flat	B-flat	C	D-flat	E-flat	F	G
E-flat	3	A-flat	B-flat	C	D	E-flat	F	G
B-flat	2	A	B-flat	C	D	E-flat	F	G
F	1	A	B-flat	C	D	E	F	G
C	0	A	B	C	D	E	F	G
G	1	A	B	C	D	E	F#	G
D	2	A	B	C#	D	E	F#	G
A	3	A	B	C#	D	E	F#	G#
E	4	A	B	C#	D#	E	F#	G#
B	5	A#	B	C#	D#	E	F#	G#
F#	6	A#	B	C#	D#	E#	F#	G#
C#	7	A#	B#	C#	D#	E#	F#	G#

Often the sequence in the first column of Table 4.1 is memorized by musicians as the "Circle of Fifths" or "Circle of Fourths" (you increase by fifths going down the column, and increase by fourths going up the column). There are different rules that you can memorize to figure out what sharps or flats are in a key. The phrase "Circle of Fifths" suggests that a graph approach might also be useful.

Key signatures that are closely related will have similar numbers of sharps or flats. Something to be careful of, however, is that some notes can be written two different ways, e.g. as A# or B-flat. It turns out that because of this, C# and A-flat only differ by one note. If we re-write the key of C# using flats, we can write it as the key of D-flat, which has the notes

A-flat B-flat C D-flat E-flat F G-flat

and this disagrees with A-flat in only one note. Consequently, the graph of closeness for keys ends up being a closed circle, unlike the graph for the modes.

Understanding which keys are most closely