COMMUTATIVE ALGEBRA, MATH 530
HOMEWORK

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(1) Let R be a commutative ring and $x \in R$. Show $x \in \text{rad}(R)$ if and only if $1 + rx$ is a unit for all $r \in R$, where $\text{rad}(R)$ is the Jacobson radical of R.
(2) Let R be a ring and I be an ideal contained in $\text{nil}(R)$. Show that if $r + I$ is a unit in R/I then r is a unit of R.
(3) Let R and S be commutative rings and $f : R \to S$ be a surjective homomorphism. Prove that $f(\text{rad}(R)) \subseteq \text{rad}(S)$. Give and example where the inclusion is strict.
(4) Let $C = \{P_j \mid j \in \mathcal{J}\}$ be a collection of prime ideals in a commutative ring R. If C forms a chain then show $\bigcap_{j \in \mathcal{J}} P_j$ is a prime ideal. Additionally prove that for every proper ideal I, there are minimal elements among the primes containing I.
(5) Let R be a commutative ring, I, $P_1, \dots P_n$ ideals of R with P_i prime for $3 \leq i \leq n$. Show that if I is not contained in any of the P_i then there exists an element $x \in I$ not contained in any P_i.
(6) Let R be a commutative ring. Prove that if $R^n \cong R^m$ then $n = m$.
(7) Let I be an ideal of a commutative ring R and M a finite R-module. Show $\sqrt{\text{ann}(M/IM)} = \sqrt{(\text{ann}(M) + I)}$.
(8) Let R be a commutative ring. If $0 \to K \to M \to N \to 0$ is an exact sequence of R-modules and K and N are finitely generated then M is finitely generated.
(9) Let R be a commutative ring and M and N be R-submodules of an R-module L. Show that if $M + N$ and $M \cap N$ are finitely generated then so are M and N.
(10) Let R be a commutative ring. If $0 \to K \to M \to N \to 0$ is an exact sequence of R-modules and K and N are finitely presented then M is finitely presented.
(11) Prove that if $0 \to M_1 \to M_2 \to \cdots \to M_n \to 0$ is an exact sequence of R-modules of finite length, then $\sum_{i=1}^{n} (-1)^i \ell(M_i) = 0$.
(12) Suppose $\{I_j\}_{1 \leq j \leq n}$ are ideals of a commutative ring R and $\bigcap_{j=1}^{n} I_j = (0)$. Show that if R/I_j are Noetherian, then R is Noetherian.
(13) If R is a Noetherian ring, then any finitely generated module is finitely presented.
(14) If R is Noetherian and $\phi \in \text{Hom}_R(R, R)$ is a surjective homomorphism then ϕ is injective.
(15) Let (R, m) be a local ring with $m = (t)$ a principal ideal. Show that if $\bigcap_{n=1}^{\infty} m^n = (0)$ then R is Noetherian and every ideal is a power of m.
(16) Show that an ideal I of a commutative ring R which contains m^n for some maximal ideal m and some natural number n is a primary ideal.
(17) Show that $P^{(n)} = P^n R_P \cap R$ is a primary ideal with radical P. Consider $R = k[[x, y]]/(xy)$ and $P = (x)$, determine $P^{(n)}$ for all natural numbers n.
(18) Let S be a multiplicative set of a commutative ring R, what is Spec R_S? Justify, your answer.
(19) Prove that for any commutative ring R, every open cover of Spec R has a finite subcover. We say that Spec R is *quasi-compact*.

(20) A topological space is Noetherian if it satisfies d.c.c. Show that if R is a Noetherian ring, then Spec R is Noetherian.
(21) Let S be the set of all nonzero divisors of a commutative R, prove that S is the largest subset of R satisfying $f : R \to R_S$ is injective and that every element of R_S is either a unit or a zero divisor.
(22) For any R-module M define the torsion of M, $T(M) = \{x \in M \mid \text{ann } x \neq 0\}$. Show that for any multiplicative set S, $T(M)_S = T(M_S)$.
(23) Prove that if $f : M \to N$ is an R-module homomorphism, show that $f(T(M)) \subseteq T(N)$ and if $0 \to K \to M \to N$ is exact then $0 \to T(K) \to T(M) \to T(N)$ is exact.
(24) We say that M is torsionfree if $T(M) = 0$. Is $U_{TF} = \{P \in \text{Spec } R \mid M_P \text{ is torsionfree}\}$ an open set? Justify your answer.
(25) Prove that for any prime of $k[x_1, \ldots, x_n]$ satisfies $\text{ht}P + \text{coht}P = \text{dim}R$.
(26) Find the associated primes of the $\mathbb{Z}$-module $\mathbb{Z} \oplus \mathbb{Z}/120\mathbb{Z}$.
(27) If M is a finite module over a Noetherian ring R and M_1 and M_2 are submodules of M with $M = M_1 + M_2$, is it true that $\text{Ass}(M) = \text{Ass}(M_1) \cup \text{Ass}(M_2)$?
(28) If R is a Noetherian ring and x is a R-regular element which is not a unit, show that the $\text{Ass}(R/xR) = \text{Ass}(R/x^n)$ for any positive integer n.
(29) Let I and J be ideals of a Noetherian ring R. Prove that if $JR_P \subseteq IR_P$ for all $P \in \text{Ass}(R/I)$ then $J \subseteq I$.
(30) Give two primary decompositions of $(x^n, x^m y) \subseteq k[x, y]$ where $n > m$. Justify your answer.
(31) Let R be a commutative ring and M and R-module. Show that M is a flat R-module if and only if M_P is a flat R_P module for all $P \in \text{Spec}R$.
(32) Let R be a Noetherian ring and P and associated prime of R. Show there is an integer c such that $P \in \text{Ass}(R/I)$ for every ideal $I \subseteq P^c$.
(33) Let $R \subseteq T$ be commutative rings with T integral over R. If $\mathfrak{p}$ is a prime of R and T has only one prime P lying over $\mathfrak{p}$, then $T_P = T_{\mathfrak{p}}$.
(34) Let $R \subseteq T$ be commutative rings with T integral over R. Show that $\text{dim}R = \text{dim}T$.
(35) Let R be a domain with field of fractions K. We say $x \in K$ is almost integral over R if there exists $c \neq 0$ in R with $cx^n \in R$ for all $n > 0$. Prove that if $x \in K$ is integral over R then x is almost integral over R. Further prove that if R is Noetherian, then the converse holds.
(36) Let $R \subseteq T$ be commutative rings with T integral over R. If P is a prime ideal of T and $\mathfrak{p} = P \cap R$, then $\text{ht}P \leq \text{ht}\mathfrak{p}$.
(37) Let R, T be commutative rings with T anr R-algebra such that the going down theorem holds between R and T. If P is a prime ideal of T and $\mathfrak{p} = P \cap R$, then $\text{ht}P \geq \text{ht}\mathfrak{p}$.
(38) Prove that in a valuation ring any finitely generated ideal is principal.
(39) If R is a valuation ring of dimension 1 and K is its field of fractions then there do not exist any intermediate rings between R and K. Conversely, show that if R is not a field and it is maximal among proper subrings, then R is a valuation ring of dimension 1.
(40) Let R be a DVR and K its field of fractions. If L is a finite extension of K, then a valuation ring of L dominating R is a DVR.
(41) Prove any ideal in a Dedekind ring can be generated by at most 2 elements.
(42) Let R be the integral closure of $\mathbb{Z}$ in $\mathbb{Q}[\sqrt{10}]$. Show that R is Dedekind but not a PID.