401 HW # 32.3, 32.5 (e, k), 33.6, 33.9, 33.11, 33.13, 34.9, 34.10, 35.3, 35.7, 35.10, 35.14, 35.15

32.3 a) Suppose $\sum_{n=1}^{\infty} a_n$ is convergent, $m \in \mathbb{N}$, $m > 1$. Prove $\sum_{n=m}^{\infty} a_n$ is convergent and

$$\sum_{n=m}^{\infty} a_n = \sum_{n=1}^{\infty} a_n - (a_1 + a_2 + \dots + a_{m-1})$$

Let $S_n = \sum_{i=1}^{n} a_i$, $t_n = \sum_{i=m}^{n} a_i$, then $S = \lim_{n\to\infty} S_n$

And $T = \lim_{n\to\infty} \sum_{i=m}^{n} a_i$; assume $n \geq m$

$$t_n = \sum_{n=1}^{n} a_i - \sum_{i=1}^{m} a_i = (a_1 + a_2 + \dots + a_n) - (a_1 + a_2 + \dots a_m)$$

$$t_n = S_n - (a_1 + a_2 + a_3 + \dots a_{m-1})$$

$$\lim t_n = \lim S_n - (a_1 + a_2 + a_3 + \dots + a_{m-1})$$

$$\sum_{n=m}^{\infty} a_n = \lim t_n = \sum_{n=1}^{\infty} a_n - (a_1 + a_2 + a_3 + \dots + a_{m-1})$$

b) Suppose $m \in \mathbb{N}$ with $m > 1$ and $\sum_{n=m}^{\infty} a_n$ convergent, if $a_1, \dots, a_{m-1}$ are real, prove $\sum_{n=1}^{\infty} a_n$ is convergent and

$$\sum_{n=1}^{\infty} a_n = a_1 + \dots + a_{m-1} + \sum_{n=m}^{\infty} a_n$$

This follows from the last example, algebraically moving $(a_1 + a_2 + \dots a_{m-1})$ to the other side of the equation to get $\sum_{n=m}^{\infty} a_n + (a_1 + \dots + a_{m-1}) = \sum_{n=1}^{\infty} a_n$

32.5 e) $\sum_{n=2}^{\infty} \frac{1}{n(n-1)}$ $\sum_{n=1}^{\infty} \frac{A}{n} + \frac{B}{n-1}$ $An - A + Bn = 1$

$A = -1$, $B = 1$

$$\sum_{n=2}^{\infty} \frac{1}{n-1} - \frac{1}{n} = \left(1 - \cancel{\tfrac{1}{2}}\right) + \left(\cancel{\tfrac{1}{2}} - \cancel{\tfrac{1}{3}}\right) + \left(\cancel{\tfrac{1}{3}} - \cancel{\tfrac{1}{4}}\right) + \dots$$

$$\sum_{n=2}^{\infty} \frac{1}{n(n-1)} = \sum_{n=1}^{\infty} \left(\frac{1}{n-1} - \frac{1}{n}\right) = 1$$

The series converges to 1

k) $\sum_{n=1}^{\infty} \frac{1}{(n)(n+1)(n+2)} = \sum_{n=1}^{\infty} \frac{A}{n} + \frac{B}{n+1} + \frac{C}{n+2}$

$$A(n+1)(n+2) + B(n)(n+2) + C(n)(n+1)$$

$$A(n^2+3n+2) + B(n^2+2n) + C(n^2+n) = 1$$

$$A+B+C=0$$
$$3A+2B+C=0$$
$$2A=1 \quad ; \quad A=\frac{1}{2}$$

$$B = -\frac{1}{2} - C$$

$$\frac{3}{2} + 2\left(-\frac{1}{2} - C\right) + C = 0$$
$$\frac{3}{2} - 1 - 2C + C = 0$$
$$C = \frac{1}{2} \qquad C = \frac{1}{2},\ B = 1$$

$$\sum_{n=1}^{\infty} \frac{1}{2}\left(\frac{1}{n}\right) - \frac{1}{n+1} + \frac{1}{2}\left(\frac{1}{n+2}\right)$$

$$= \sum_{n=1}^{\infty} \frac{1}{2}\left(\frac{1}{n} - \frac{1}{n+1}\right) - \sum_{n=1}^{\infty} \frac{1}{2}\left(\frac{1}{n+1} - \frac{1}{n+2}\right)$$

$$= \lim_{n\to\infty} \frac{1}{2} - \frac{1}{2(n+1)} - \lim_{n\to\infty}\left(\frac{1}{4} - \frac{1}{2(n+2)}\right)$$

$$= \frac{1}{2} - \frac{1}{4} = \frac{1}{4}$$

33.6 If Σa_n and Σb_n converges, $\Sigma(a_n b_n)$ does not necessarily converge

let $a_n = b_n = \frac{(-1)^n}{\sqrt{n}}$ convergent by AST, $\sum_{n=1}^{\infty} a_n b_n = \frac{1}{n}$, divergent by harmonic series

33.9 $|a_{n+1} - a_n| \leq b_n$, $\sum_{n=1}^{\infty} b_n$ is convergent, show (a_n) converges

Since $\sum_{n=1}^{\infty} b_n$ is convergent, $\lim_{n\to\infty} b_n = 0$, let $\sum_{n=1}^{\infty} b_n = S$

Therefore, $\lim_{n\to\infty} |a_{n+1} - a_n| = 0$

$S = \lim_{n\to\infty} \sum_{k=1}^{n} (a_{k+1} - a_k) \leq \sum_{k=1}^{n} b_k$; $(\cancel{a_2} - a_1) + (\cancel{a_3} - \cancel{a_2}) + (\cancel{a_4} - \cancel{a_3}) + \ldots + (\cancel{a_n} - \cancel{a_{n-1}}) + (a_{n+1} - \cancel{a_n})$

$\lim_{n\to\infty} a_{n+1} - a_1 = S$

$\lim_{n\to\infty} a_{n+1} = S + a_1$

$\therefore$ a_{n+1} is convergent, a_n is convergent

33.11 $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{2^2} + \frac{1}{5} - \frac{1}{2^3} + \frac{1}{7} - \frac{1}{2^4} \ldots$ diverges

This does not contradict the AST, since the terms are not decreasing $(\frac{1}{2^3} = \frac{1}{8} < \frac{1}{7})$.

33.13 since $a_k \geq a_{k+1}$ then $2a_2 \geq a_2 + a_3$ and $4a_4 \geq a_4 + a_5 + a_6 + a_7, \ldots, 2^k a_{2^k} \geq a_{2^k} + \cdots + a_{2^{k+1}-1}, \ldots$

so $\sum_{k=0}^{\infty} 2^k a_{2^k} \geq \sum_{n=0}^{\infty} a_n$

If $\sum_{k=0}^{\infty} 2^k a_{2^k}$ converges, then $\sum_{n=0}^{\infty} a_n$ is bounded above & since $S_n = \sum_{m=0}^{n} a_m$ increasing then $\sum_{n=0}^{\infty} a_n$ converges.

For $\varepsilon > 0$ $\exists N$ s.t. $|a_m + \cdots + a_n| < \varepsilon/2$ $\forall n \geq m > N$ if $\sum_{n=0}^{\infty} a_n$ converges. Suppose $2^k > N$

$ consider $|2^{k+1} a_{2^{k+1}} + \cdots + 2^r a_{2^r}| < 2|a_{2^k} + \cdots + a_{2^{k+1}-1} + \cdots a_{2^{r-1}} + \cdots + a_{2^r-1}|$

$< 2 \cdot \varepsilon/2$

which implies $\sum_{k=0}^{\infty} 2^k a_{2^k}$ converges by the Cauchy criterion

b) Show $\sum_{n=1}^{\infty} 1/n^p$ converges if $p > 1$, diverges if $0 < p \leq 1$

$\sum 1/n^p$ converges iff $\sum 2^k \cdot 1/2^{kp}$ converges

$\sum \frac{1}{2^{k(p-1)}} = \sum 2^{(1-p)k}$

By Geometric Series, this converges if $|2^{(1-p)k}|$

$\log_2(2^{(1-p)}) < \log_2 1$, $1-p < 0$, $p > 1$. Therefore, this converges iff $p > 1$

c) Show $\sum_{n=2}^{\infty} \frac{1}{n(\ln(n))^p}$ converges if $p > 1$, diverges if $0 < p \leq 1$

by a, b, show $\sum_{k=0}^{\infty} 2^k \frac{1}{2^k(\ln(2^k))^p}$ converges

$\sum \cancel{2^k} \frac{1}{\cancel{2^k}\ln(2^k))^p} = \sum \frac{1}{\ln 2^p} \frac{1}{k^p} = \frac{1}{\ln 2^p} \sum \frac{1}{k^p}$ (since p is a constant)

By b), this converges if $p > 1$, diverges if $0 < p \leq 1$

34.9 Suppose $\sum_{n=0}^{\infty} a_n x^n$ has radius of convergence 2.
Find R for

a) $\sum_{n=0}^{\infty} a_n{}^k x^n$

$\lim \left|\frac{a_{n+1}}{a_n}\right| = \frac{1}{2} = \lim |a_n|^{1/n}$

$\lim |a_n{}^k|^{1/n} = \lim |a_n|^{k/n} = \left(\lim |a_n|^{1/n}\right)^k = \left(\frac{1}{2}\right)^k$

$R = \frac{1}{\lim |a_n|^{1/n}} = \left(\frac{1}{2}\right)^{-k} = 2^k$, The radius of convergence is 2^k.

b) let $\sum_{n=0}^{\infty} a_n x^{kn} = \sum_{n=0}^{\infty} a_n (x^k)^n$, therefore this series converges if $|x^k| < 2$ or $|x| < 2^{1/k}$, therefore, the radius of $2^{1/k}$.

c) $\sum a_n x^{n^2} = \sum a_n (x^2)^n$, Therefore this series converges if $|x^2| < 2$, or $|x| < \sqrt{2}$, The radius for convergence is $\sqrt{2}$

34.10 Prove $\sum_{n=0}^{\infty} a_n x^n$ and $\sum_{n=0}^{\infty} n a_n x^n$ have the same radius of convergence

let $b_n = n a_n$, then $\sum a_n x^n$ has the radius of convergence $\lim\left(\frac{a_n}{a_{n+1}}\right) = R$, and $\sum b_n x^n$ has the radius of convergence $\lim\left(\frac{b_n}{b_{n+1}}\right) = \lim\left(\frac{n a_n}{(n+1) a_n}\right)$
$= \lim\left(\frac{a_n}{(1+\frac{1}{n}) a_{n+1}}\right)$. If $\lim a_n$ is finite, this limit equals $\left(\frac{a_n}{a_{n+1}}\right) = R$

if a_n is infinite, the radius of convergence must be 0, (x^n must be 0), which would give both series a radius of convergence of 0.

35.3 Let $f_n(x) = x^n/n$ for $x \in [-1,1]$. Find $f(x) = \lim f_n(x)$ and find if it conforms uniformly.

$\forall x \in [-1,1]$, $\left|\frac{x^n}{n}\right| \leq \frac{1}{n}$, and $\lim_{n\to\infty} \frac{1}{n} = 0$,

Therefore, this function converges uniformly to $f(x) = 0$ $\forall x \in [-1,1]$

35.7 Let $f_n(x) = 1/(1+x^n)$ for $x \in [0,1]$

If $x \neq 1$, $\lim x^n \to 0$, $f_n(x) \to 1$

If $x = 1$, $\lim x^n \to 1$, $f_n(x) \to \frac{1}{1+1} = \frac{1}{2}$

b) $f_n(x)$ converges uniformly on $[0,t)$ $0<t<1$

$\forall \varepsilon > 0$ $\exists N \in \mathbb{N}$ s.t. for all $x \in [0,t)$ $n > N$ implies $|f(x) - f_n(x)| < \varepsilon$ Note, $\exists N$ st. $\forall n > N$ $|x^n| < \varepsilon$

~~[illegible]~~

~~[illegible]~~

But $|f(x) - f_n(x)| = \left|1 - \frac{1}{1+x^n}\right| = \left|\frac{x^n}{1+x^n}\right| < |x^n| < \varepsilon$

So $f_n(x)$ converges uniformly.

c) let $\varepsilon = \frac{1}{4}$, $\exists\, N$ such that $\left|\frac{x^n}{1+x^n} - 1\right| < \frac{1}{4}$ $\forall n > N$ for $0 \le x < 1$. Thus $\frac{x^n}{1+x^n} > 3/4$ and $\left|\frac{x^n}{1+x^n} - \frac{1}{2}\right| > 3/4 - \frac{1}{2} = \frac{1}{4}$.

Thus f_n does not converse uniformly to f on $[0,1]$.

35.10 If $f_n \longrightarrow f$ uniformly then $\forall \varepsilon > 0$ $\exists N$ s.t. $\forall n > N$

$|f_n(x) - f(x)| < \varepsilon$ $\forall x \in S$. $\Longrightarrow$

$\sup\limits_{x \in S} |f_n(x) - f(x)| < \varepsilon$ which implies

$\lim\limits_{n \to \infty} \sup\limits_{x \in S} |f_n(x) - f(x)| = 0$

clearly $\sup\limits_{x \in S} |f_n(x) - f(x)| \ge |f_n(x) - f(x)|$

so if $\lim\limits_{n \to \infty} \sup\limits_{x \in S} |f_n(x) - f(x)| = 0$, $\forall \varepsilon > 0$

$\exists N$ s.t. $\forall n > N$ then $\sup\limits_{x \in S} |f_n(x) - f(x)| < \varepsilon$

$\Longrightarrow |f_n(x) - f(x)| < \varepsilon$ $\forall x$

and $f_n(x)$ converges to $f(x)$ uniformly

Since this is a lim sup, $|f(x)-f_n(x)| \leq \lim_{n\to\infty}[\sup\{|f(x)-f_n(x)| : x\in S\}] < \varepsilon$, $f(x)$ converges uniformly.

35.14 If f_n and g_n converge uniformly on S, prove f_n+g_n converges uniformly on S.

Since f_n and g_n converge uniformly, $\forall \varepsilon>0$, $\exists N\in\mathbb{N}$ st. $n>N$ $\forall x\in S$ implies

$$|f(x)-f_n(x)| < \varepsilon/2 \quad \text{and} \quad |g(x)-g_n(x)| < \frac{\varepsilon}{2}$$

$$|(f(x)+g(x))-(f_n(x)+g_n(x))| \underset{\text{By Triangle inequality.}}{\leq} |f(x)-f_n(x)|+|g(x)-g_n(x)| < \frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon$$

Therefore, f_n+g_n converges uniformly on S.

35.15 Suppose f_n converges uniformly to f on S and f_n is bounded on S $\forall n\in\mathbb{N}$

a) prove f is bounded on S.

Since f_n converges uniformly, $\forall\varepsilon>0, \exists N\in\mathbb{N}$ st $n>N$ implies $\forall x\in S$, $|f(x)-f_n(x)|<\varepsilon$;

$$|f(x)| \underset{\text{by triangle}}{\leq} |f(x)-f_n(x)|+|f_n(x)| < \varepsilon+|f_n(x)|$$

Therefore, $|f(x)|$ is bounded by $|f_n(x)|$

b) Prove the sequence $f_n(x)$ is uniformly bounded on S. ie. $\exists$ M s.t. $|f_n(x)| \leq M \ \forall \ x \in S, \ n \in \mathbb{N}$

By a, $f(x)$ is bounded, Therefore, let $|f(x)| < K$

By the previous argument (a), $\exists N$ st. $n > N$ implies $|f_n(x) - f(x)| < \varepsilon$

$$|f_n(x)| \leq |f_n(x) - f(x)| + |f(x)| < \varepsilon + |f(x)| < \varepsilon + K$$

Therefore, all terms in $f_n(x)$ are bounded by K, and $f_n(x)$ is uniformly bounded.

c) Find a sequence of bounded functions that converge pointwise to an unbounded function.

~~$f_n(x) = 0$ if $0 < x < \frac{1}{n}$, $f_n(x) = \frac{1}{nx}$ if $\frac{1}{n} \leq x \leq 1$~~

~~$\lim_{n \to \infty} f_n(x) = \frac{1}{n}(x)$ for $[0,1]$~~

$$f_n(x) = \begin{cases} x^2 & |x| \leq n \\ n^2 & |x| > n \end{cases}$$

$f_n(x) \longrightarrow x^2$ but $f_n(x)$ bdd.