Homework #1

3.1

(a) FALSE
p is referred to as the hypothesis

(b) FALSE
The contrapositive of $p \Rightarrow q$ is $\sim q \Rightarrow \sim p$

(c) FALSE
The inverse of $p \Rightarrow q$ is $p \wedge \sim q$.

(d) FALSE
Since it has to be true for all possible values of n, one example is not enough.

(e) TRUE
It is enough to find one n that complies with $p(n)$.

3.6

(a) Consider $x = -3$. We have $x^2 = 9 > 4$, however $x = -3 < 2$. ✓

(b) Consider $n = 41$. This yields $n^2 + n + 41 = 1763$, which is divisible by 41 ($1763/41 = 43$), so it is not prime. ✓

(d) Consider 113. I checked that it's prime by dividing it by every prime number less than $113/2 = 56.5$; the result never was an integer, so it is prime. ✓

(g) Take $x = 18$, $y = 2$. Clearly, they are unequal integers also notice that $xy = 36 = 6^2$ is a perfect square. However, neither x or y are perfect squares. ✓

(i) Consider a real number $x < 1$, e.g. $x = 0.5$. Even though $x > 0$, we don't have $x^2 < x^3$ because $x^2 = 0.25 > x^3 = 0.125$. ✓

3.7 p and q are integers.
An integer m is even iff $m = 2k$ for some integer k.
An integer m is odd iff $m = 2k+1$ for some integer k.
Prove

(a) If p is odd and q is odd, then $p+q$ is even.

Proof: Since p is odd, we can rewrite it as $p = 2k_1 + 1$, likewise q is odd, so it can be rewritten as $q = 2k_2 + 1$.

Now,
$$\begin{aligned} p+q &= 2k_1 + 1 + 2k_2 + 1 \\ &= 2k_1 + 2k_2 + 2 \\ &= 2(k_1 + k_2 + 1) \end{aligned}$$

But $k_1 + k_2 + 1$ is an integer, which can be called k_3, so we have

$$p+q = 2k_3$$

which is even because an integer m is even iff $m = 2k$ for some integer k. $\square$

(b) If p is odd and q is odd, then pq is odd.

Proof: Since both p and q are odd they can be rewritten as $p = 2k_1 + 1$ and $q = 2k_2 + 1$. So,

$$\begin{aligned} pq &= (2k_1 + 1)(2k_2 + 1) \\ &= 4k_1k_2 + 2k_1 + 2k_2 + 1 \end{aligned}$$

Now,
$$\begin{aligned} 2k_1 + 2k_2 + 1 &= 2(k_1 + k_2) + 1 \\ &= 2k_3 + 1 \end{aligned}$$

which is an odd number (see statements at the top of the page).
On the other hand,
$$\begin{aligned} 4k_1k_2 &= 2 \cdot (2k_1k_2) \\ &= 2k_4 \end{aligned}$$

is even. So pq consists of the addition of an even number with an odd number, which results in an odd number:

$$\begin{aligned} pq &= 2k_4 + (2k_3 + 1) \\ &= 2(k_3 + k_4) + 1 \\ &= 2k_5 + 1 \end{aligned}$$

i.e. pq is odd. $\square$

3-0235 — 50 SHEETS — 5 SQUARES
3-0236 — 100 SHEETS — 5 SQUARES
3-0237 — 200 SHEETS — 5 SQUARES
3-0137 — 200 SHEETS — FILLER

COMET

3.7

(c) If p is odd and q is even, then $p+q$ is odd.

Proof. Writing p as $2K_1+1$ and q as $2K_2$ we have,

$$\begin{aligned} p+q &= 2K_1+1+2K_2 \\ &= 2(K_1+K_2)+1 \\ &= 2K_3+1 \end{aligned}$$

where $K_3 = K_1+K_2$ is also an integer. Now, since a number m is odd iff $m = 2K+1$ for some K integer,

$$p+q = 2K_3+1$$

is obviously odd. $\square$

3.9 Assume given hypotheses are true, use tautologies to establish conclusion.

(a) Hypotheses $r \Rightarrow \sim s,\ t \Rightarrow s$
Conclusion $r \Rightarrow \sim t$

Proof: * Using tautology $[p \Rightarrow q \Leftrightarrow \sim q \Rightarrow \sim p]$ on the 2nd hypothesis

$$t \Rightarrow s \Leftrightarrow \sim s \Rightarrow \sim t$$

So, we have the hypotheses: $r \Rightarrow \sim s$, $\sim s \Rightarrow \sim t$

* Using tautology $[(p \Rightarrow q) \wedge (q \Rightarrow r)] \Rightarrow (p \Rightarrow r)$ we conclude:

$$r \Rightarrow \sim t$$

$\square$

(b) Hypotheses: $\sim t,\ (r \vee s) \Rightarrow t$
Conclusion $\sim s$

Proof: * Using tautology $[\sim q \wedge (p \Rightarrow q)] \Rightarrow \sim p$ we get

$$\sim t \wedge (r \vee s) \Rightarrow t \Rightarrow \sim(r \vee s)$$

By De Morgan's law we have $\sim(r \vee s) \Leftrightarrow \sim r \wedge \sim s$

* Using tautology $(p \wedge q) \Rightarrow p$ we get

$$\sim r \wedge \sim s \Leftrightarrow \sim s \wedge \sim r \Rightarrow \sim s$$

In conclusion: $\sim t \wedge (r \vee s) \Rightarrow t \Rightarrow \sim s$ $\square$

CONT...

3-0235 — 50 SHEETS — 5 SQUARES
3-0236 — 100 SHEETS — 5 SQUARES
3-0237 — 200 SHEETS — 5 SQUARES
3-0137 — 200 SHEETS — FILLER
COMET

39. CONT....

(c) Hypotheses: $r \Rightarrow \sim s$, $\sim r \Rightarrow \sim t$, $\sim t \Rightarrow u$, $v \Rightarrow s$
Conclusion $\sim v \vee u$

Proof: * Using tautology $[(p \Rightarrow q) \wedge (q \Rightarrow r)] \Rightarrow (p \Rightarrow r)$ (11)

$r \Rightarrow \sim s$, $\sim r \Rightarrow \sim t$, $\sim t \Rightarrow u$, $v \Rightarrow s$

$\Rightarrow r \Rightarrow \sim s$, $\sim r \Rightarrow u$, $v \Rightarrow s$ (*)

Taking the tautology $[(p \Rightarrow q) \Leftrightarrow [(\sim q) \Rightarrow (\sim p)]]$ (2)

$r \Rightarrow \sim s \Leftrightarrow s \Rightarrow \sim r$, so we have

$s \Rightarrow \sim r$, $\sim r \Rightarrow u$, $v \Rightarrow s$

Reapplying tautology (11)

$s \Rightarrow \sim r$, $\sim r \Rightarrow u$, $v \Rightarrow s \Rightarrow [s \Rightarrow u, v \Rightarrow s]$

but $[s \Rightarrow u, v \Rightarrow s] \Leftrightarrow [v \Rightarrow u]$

But $v \Rightarrow u$ is logically equivalent to $(\sim v \vee u)$
(see table below)

u	v	$\sim v$	$v \Rightarrow u$	$\Leftrightarrow$	$\sim v \vee u$
0	0	1	1	1	1
0	1	0	0	1	0
1	0	1	1	1	1
1	1	0	1	1	1

So, we can conclude,

$(r \Rightarrow \sim s, \sim r \Rightarrow \sim t, \sim t \Rightarrow u, v \Rightarrow s) \Rightarrow \sim v \vee u$

$\square$

4.4 Prove: There exists a rational number n s.t. $n^2 + \frac{3}{2}n = 1$. Is it unique?

Proof It is enough to find one n which satisfies the given equation. Finding that n can be done by solving the equation:

$$n^2 + \frac{3}{2}n - 1 = 0$$

Which can be solved using the quadratic formula

$$n_{1,2} = \frac{1}{2}\left(-\frac{3}{2} \pm \sqrt{\frac{9}{4} - 4\cdot(-1)}\right)$$

$$= \frac{1}{2}\left(-\frac{3}{2} \pm \sqrt{\frac{25}{4}}\right)$$

$$= \frac{1}{2}\left(-\frac{3}{2} \pm \frac{5}{2}\right)$$

$$n_1 = \frac{1}{2}$$

$$n_2 = -2$$

So by taking $n = \frac{1}{2}$, we have $n^2 + \frac{3}{2}n = 1$ and since n is rational, the statement is true.

□

The choice of n is not unique because $n = 2$ also satisfies the equation and is also rational.

4.8. Prove. If $\frac{x}{x-1} \leq 2$, then $x < 1$ or $x \geq 2$.

Proof: We will prove the contrapositive:

If $x \geq 1$ and $x < 2$, then $\frac{x}{x-1} > 2$

For the inequality $\frac{x}{x-1} > 2$, to make sense, we need to exclude the case when $x = 1$. So we will first prove that if $x > 1$ and $x < 2$ then $\frac{x}{x-1} > 2$

CONTINUES...

4.8 CONT...

We have $x > 1$ and $x < 2$

Starting from $x < 2$

we can add x on both sides:

$$x + x < 2 + x$$
$$2x < 2 + x$$
$$2x - 2 < x$$
$$2(x-1) < x$$

Since $x > 1 \Rightarrow x - 1 > 0$, we can divide by $x-1$ and the direction of the inequality doesn't change:

$$2 < \frac{x}{x-1}$$

So, we just showed that if $x > 1$ and $x < 2 \Rightarrow \frac{x}{x-1} > 2$

Now, for the case when $x = 1$, we can analyze the limit:

$\lim_{x \to 1^+} \frac{x}{x-1}$, which complies with the inequality.

So, if we take the contrapositive, we proved that

If $\frac{x}{x-1} \leq 2 \Rightarrow x < 1$ or $x \geq 2$.

$\square$

4.11 What is wrong with each of those "proofs"?

(a) The problem with this "proof" is that it's starting from the assumption that the conclusion is true.
The correct way of doing it would be to assume that m^2 is odd and then conclude that m is odd.
Another valid way would be to prove the contrapositive i.e. $\sim q \Rightarrow \sim p$. So it shoud be assumed that m is not odd and then conclude that m^2 is not odd.

(b) Here, the contrapositive is being proved.
Everything is OK.

4.17 Prove or give a counterexample There do not exist 3 consecutive even integers a, b, c such that $a^2+b^2=c^2$

Sol: Let's find a counterexample,

If such integers exist and they are consecutive, one should be able to rewrite them as:

$a = 2k \qquad b = 2k+2 \qquad c = 2k+4$

Let's find the value of k s.t. $a^2+b^2=c^2$

$$(2k)^2 + (2k+2)^2 = (2k+4)^2$$
$$4k^2 + 4k^2+8k+4 = 4k^2+16k+16$$
$$4k^2 - 8k - 12 = 0$$
$$k_{1,2} = \frac{1}{2\cdot 4}\left(8 \pm \sqrt{8^2-4\cdot 4\cdot(-12)}\right)$$
$$k_1 = 3$$
$$k_2 = -1$$

So, by taking $a=6$, $b=8$, $c=10$, we have
$$6^2+8^2 = 100 = 10^2$$
which is a counterexample, so the statement is false. □

4.19 Prove or give a counterexample.
The sum of any 5 consecutive integers is divisible by 5.

We will try to prove that if a, b, c, d, e are consecutive integers, then

$$a+b+c+d+e = 5k \quad , \text{ for some } k \text{ integer.}$$

Proof. Since a, b, c, d, e are consecutive, we can rewrite them as

$$a = a \qquad d = a+3$$
$$b = a+1 \qquad e = a+4$$
$$c = a+2$$

So, the sum becomes:

$$\begin{aligned} a+b+c+d+e &= a+(a+1)+(a+2)+(a+3)+(a+4) \\ &= 5a+10 \\ &= 5(a+2) \end{aligned}$$

Since a is an integer, $(a+2)$ is also an integer and it can be rewritten as m (also an integer). So we have

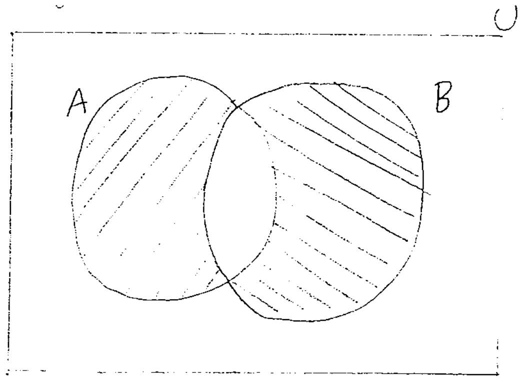
$$a+b+c+d+e = 5m \quad , \quad m \in \mathbb{Z}$$

So, the initial statement was true.

$\square$

5.7 $A \triangle B = (A \setminus B) \cup (B \setminus A)$

(a) Venn diagram for $A \triangle B$



(b) $A \triangle A$?

$$A \triangle A = (A \setminus A) \cup (A \setminus A) = \emptyset$$

(c) $A \triangle \emptyset$?

$$A \triangle \emptyset = (A \setminus \emptyset) \cup (\emptyset \setminus A) = A \cup \emptyset = A$$

(d) $A \triangle U$?

$$A \triangle U = (A \setminus U) \cup (U \setminus A) = \emptyset \cup A^c = A^c$$

5.14 Which statements would enable to conclude that $x \in A \cup B$?

(a) $x \in A$ and $x \in B$ – **YES**
Because for this statement to be true both $x \in A$ and $x \in B$ have to be true, i.e. x will be both in A and B, so $x \in A \cup B$.

(b) $x \in A$ or $x \in B$ – **YES**
Because for the statement to be true, it's enough for either $x \in A$ to be true or $x \in B$ to be true. So at least x will be either in A or B (or both). So, x will also be in the union, i.e. $x \in A \cup B$.

CONTINUES...

3-0235 — 50 SHEETS — 5 SQUARES
3-0236 — 100 SHEETS — 5 SQUARES
3-0237 — 200 SHEETS — 5 SQUARES
3-0137 — 200 SHEETS — FILLER

COMET

5.14 CONT...

(c) If $x \in A$, then $x \in B$ — NO

Because this statement will still be true when both the hypothesis and conclusion are false, i.e. $x \notin A$ and $x \notin B$. In that case we couldn't say that $x \in A \cup B$.

(d) If $x \notin A$, then $x \in B$ — YES

The three cases when this statement will be true are:

$\sim(x \notin A)$ and $\sim(x \in B)$ which is equivalent to $x \in A$ and $x \notin B$
$\sim(x \notin A)$ and $x \in B$ which is equivalent to $x \in A$ and $x \in B$
$x \notin A$ and $x \in B$

So, in all three cases x is always either in A, B, or both, so we can conclude that $x \in A \cup B$.

5.18 Prove that the empty set is unique.

Proof: Suppose A and B are empty sets.

We know that the empty set is the subset of any other set X, thus:

$$A \subseteq X \qquad \forall X \qquad (*)$$

Specifically we can take $X = B$, so we get

$$A \subseteq B.$$

However, B is also an empty set, so:

$$B \subseteq X \qquad \forall X$$

In particular X can be equal to A, so

$$B \subseteq A \qquad (**)$$

So, by $(*)$ and $(**)$ we get

$$A = B$$

In other words, the emply set is unique. $\square$

5.25 Find $\bigcup_{B\in\mathcal{B}}$ and $\bigcap_{B\in\mathcal{B}}$ for each collection $\mathcal{B}$

(a) $\mathcal{B} = \left\{ \left[1, 1+\frac{1}{n}\right] : n \in \mathbb{N} \right\}$

$$\bigcup_{B\in\mathcal{B}} = \left\{\left[1, 1+\tfrac{1}{1}\right]\right\} \cup \left\{\left[1, 1+\tfrac{1}{2}\right]\right\} \cup \left\{\left[1, 1+\tfrac{1}{3}\right]\right\} \ldots$$

$$\Rightarrow \bigcup_{B\in\mathcal{B}} = [1, 2]$$

$$\bigcap_{B\in\mathcal{B}} = \{[1, 2]\} \cap \left\{\left[1, 1+\tfrac{1}{2}\right]\right\} \ldots$$

when $n \to \infty$ we have $\lim_{n\to\infty}\left(1+\frac{1}{n}\right) = 1$

$$\Rightarrow \bigcap_{B\in\mathcal{B}} = \{1\}$$

(b) $\mathcal{B} = \left\{ \left(1, 1+\frac{1}{n}\right) : n \in \mathbb{N} \right\}$

$$\bigcup_{B\in\mathcal{B}} = \left\{\left(1, 1+\tfrac{1}{1}\right)\right\} \cup \left\{\left(1, 1+\tfrac{1}{2}\right)\right\} \cup \ldots$$

$$\bigcup_{B\in\mathcal{B}} = \{(1, 2)\}$$

$$\bigcap_{B\in\mathcal{B}} = \{(1,2)\} \cap \left\{\left(1, 1+\tfrac{1}{2}\right)\right\} \cap \ldots$$

$$\bigcap_{B\in\mathcal{B}} = \emptyset$$

3-0235 — 50 SHEETS — 5 SQUARES
3-0236 — 100 SHEETS — 5 SQUARES
3-0237 — 200 SHEETS — 5 SQUARES
3-0137 — 200 SHEETS — FILLER

COMET

6.10 Prove or give a counterexample.

(a) $A \times B = B \times A$

This is false. To see this, take the following counterexample.
Assume:
$A = \{a, b\}$
$B = \{c, d\}$

$\Rightarrow A \times B = \{(a,c), (a,d), (b,c), (b,d)\}$
$B \times A = \{(c,a), (c,b), (d,a), (d,b)\}$

Obviously $A \times B \neq B \times A$, so the statement is false.

(d) $(A \times B) \cup (C \times D) = (A \cup C) \times (B \cup D)$

This is false. Take the counterexample where the sets are defined as follows:

$A = \{a\}$ $B = \{b\}$ $C = \{c\}$ $D = \{d\}$

So $A \times B = \{(a, b)\}$ $C \times D = \{(c, d)\}$

$\Rightarrow (A \times B) \cup (C \times D) = \{(a, b), (c, d)\}$

Now $A \cup C = \{a, c\}$ and $B \cup D = \{b, d\}$

and $(A \cup C) \times (B \cup D) = \{(a,b), (a,d), (c,b), (c,d)\}$

Clearly, $(A \times B) \cup (C \times D) \neq (A \cup C) \times (B \cup D)$

so, the initial statement is false.

6.12 Find examples of relations with the following properties

(a) Reflexive but not symmetric and not transitive.

Define $\widetilde{R} = \{ (a, b) \in \mathbb{R} \times \mathbb{R} \mid a < b + 1 \}$

* It IS reflexive because for any number k: $k < k + 1$
so $k \widetilde{R} k$ always.

* It is NOT symmetric because e.g. $a = 8$, $b = 11$,
clearly $a = 8 < b + 1 = 12$, but b is not related to a:
i.e. 11 is not smaller than $8 + 1 = 10$.

CONT...

3-0235 — 50 SHEETS — 5 SQUARES
3-0236 — 100 SHEETS — 5 SQUARES
3-0237 — 200 SHEETS — 5 SQUARE
3-0137 — 200 SHEETS — FILLER

COMET

6.12 (a) CONT...

* It is NOT transitive

$8 \tilde{R}\ 7.1$ because $8 < 7.1 + 1 = 8.1$
$7.1 \tilde{R}\ 6.2$ because $7.1 < 6.2 + 1 = 7.2$

but 8 is not related to 6.2, i.e 8 is not less than $6.2 + 1 = 7.2$.

(b) Symmetric, but not reflexive and not transitive

Consider $\tilde{A} = \{ (a,b) \in \mathbb{R} \times \mathbb{R} \mid |a-b| > 1 \}$

where $|\cdot|$ is the absolute value.

* It is symmetric.

We'll show that if $a_1 \tilde{A}\ b_1$ then $b_1 \tilde{A}\ a_1$.

So, suppose $a_1 \tilde{A}\ b_1$, then $|a_1 - b_1| > 1$, which means that either

$$a_1 - b_1 > 1 \quad \text{or} \quad -(a_1 - b_1) > 1$$
$$\Leftrightarrow -b_1 + a_1 > 1 \quad \text{or} \quad b_1 - a_1 > 1$$
$$\Leftrightarrow -(b_1 - a_1) > 1 \quad \text{or} \quad b_1 - a_1 > 1$$
$$\Leftrightarrow |b_1 - a_1| > 1$$

In othe words, $b_1 \tilde{A}\ a_1$.

$\square$

* It is NOT reflexive, a_1 is not related to itself,

$|a_1 - a_1| = 0$ which is not greater than 1.

* It is NOT transitive.

Take as counterexample: $a_1 = 2$, $b_1 = 3.1$, $c_1 = 1.5$

$a_1 \tilde{A}\ b_1$ $\quad (|a_1 - b_1| = 1.1 > 1)$
$b_1 \tilde{A}\ c_1$ $\quad (|b_1 - c_1| = 1.6 > 1)$

but a_1 is not related to c_1, i.e $|a_1 - c_1| = 0.5$ is not greater than 1. $\square$

(c) Transitive, but not reflexive or symmetric

Consider. $\widetilde{B} = \{(a,b) \in \mathbb{R}\times\mathbb{R} \mid a<b\}$

* It is transitive

We will show that if $a_1 \widetilde{B}\, b_1$ and $b_1 \widetilde{B}\, c_1$ then $a_1 \widetilde{B}\, c_1$

Since $a_1 \widetilde{B}\, b_1$ we have $a_1 < b_1$, also since $b_1 \widetilde{B}\, c_1$ we have $b_1 < c_1$, so clearly

$$a_1 < c_1 \Rightarrow a_1 \widetilde{B}\, c_1$$

$\square$

* It is not reflexive

a_1 is never related to itself because $a_1 = a_1$

* It is not symmetric

For example: $a_1 = 5$, $b_1 = 10$, clearly $a_1 \widetilde{B}\, b_1$ but b_1 is not related to a_1 $(10 > 5)$.

6.14 * $(a,b)\, R\, (c,d)$ iff $a+d = b+c$. Verify that R is an equivalence relation.

* Reflexive $\quad (a,b) \overset{?}{R} (a,b)$

$$a+b = a+b \ \checkmark \quad \Rightarrow \quad (a,b)\, R\, (a,b)$$

* Symmetric. If $(a,b)\, R\, (c,d)$ show that $(c,d)\, R\, (a,b)$

$$(a,b)\, R\, (c,d) \Rightarrow a+d = b+c \iff b+c = d+a$$
$$\iff c+b = d+a$$

$$\Rightarrow (c,d)\, R\, (a,b)$$

* Transitive: If $(a,b)\, R\, (c,d)$ & $(c,d)\, R\, (e,f)$

$$(**) \quad a+d = b+c \quad \& \quad c+f = e+d \quad (*)$$

From $(*)$ we have $c = e+d-f$, substituting in $(**)$

CONT...

6.14 CONT...

$$a+\cancel{d} = b+e+\cancel{d}-f \iff a+f = b+e$$

So $(a,b) \, R \, (e,f)$

$\square$

* Describe the equivalence class $E_{(7,3)}$

$E_{(7,3)} = \{ (c,d) \mid (7,3) \, R \, (c,d) \}$, i.e. it's the set s.t.

$$7+d = 3+c$$
$$\iff 4 = c-d$$

So, $E_{(7,3)} = \{ (c,d) \mid c-d = 4 \}$, i.e. all the real numbers that have their difference (first − second) equal to 4.

6.21 Relation on $\mathbb{Z}$: xRy iff $x+y = 2k$, $k \in \mathbb{Z}$
Is R an eq. relation?

* Reflexive: $x \overset{?}{R} x$

$x+x = 2x$ and since $x \in \mathbb{Z} \Rightarrow x+x = 2k$, $k = x$

So it is reflexive.

* Symmetric: If xRy show that yRx

xRy, so $x+y = 2k$

$y+x = 2k \Rightarrow yRx$. $\Rightarrow$ it is symmetric

* Transitive. If xRy & yRz we have,

(**) $x+y = 2k$ & $y+z = 2l$ (*), $k,l \in \mathbb{Z}$

From (**), $x = 2k-y$ and from (*) $z = 2l-y$
adding them up: $x+z = 2k-y+2l-y = 2(k+l-y)$

but k,l,y are all integers $\Rightarrow k+l-y$ is also an integer.
$\Rightarrow x+z = 2m$, $m \in \mathbb{Z} \Rightarrow xRz$ i.e. R is transitive.

$\Rightarrow$ Since R is symmetric, reflexive and transitive; it is an equivalence relation $\square$

3-0235 — 50 SHEETS — 5 SQUARES
3-0236 — 100 SHEETS — 5 SQUARES
3-0237 — 200 SHEETS — 5 SQUARES
3-0137 — 200 SHEETS — FILLER

COMET

6.25 R on $\mathbb{Z} \times (\mathbb{Z} \setminus \{0\})$, $(a,b) R (x,y)$ iff $ay = bx$

(a) Prove that R is an equivalence relation

* Reflexive $\quad (a,b) \overset{?}{R} (a,b)$

$$ab = ba \quad \Rightarrow \quad (a,b) R (a,b)$$

* Symmetric: If $(a,b) R (c,d)$ show $(c,d) R (a,b)$

$(a,b) R (c,d)$, so $\quad ad = bc \iff cb = da$

$$\Rightarrow (c,d) R (a,b)$$

* Transitive

If $(a,b) R (c,d)$ and $(c,d) R (e,f)$, then

(**) $ad = bc$ & $cf = de$ (*)

Since $f \neq 0$, from (*) we get $c = \frac{de}{f}$, substituting in (**)

$$a\cancel{d} = b \cdot \frac{\cancel{d}e}{f} \iff af = be \iff (a,b) R (e,f)$$

Since R is reflexive, symmetric and transitive, it is an equivalence relation $\square$

(b) Describe the equivalence classes corresponding to R.

The equivalence class are all the numbers (x,y) such that

$$\frac{a}{b} = \frac{x}{y}.$$

6.27 R, S relations on a set A. Prove or give a counterexample.

(b) If R and S are reflexive, then $R \cup S$ is reflexive.

Proof: If R and S are reflexive, then aRa and aSa, for $a \in A$, or in other words,

$$\left.\begin{matrix} \{(a,a) \mid a \in A\} \subseteq R \\ \{(a,a) \mid a \in A\} \subseteq S \end{matrix}\right\} \Rightarrow \{(a,a) \mid a \in A\} \subseteq R \cup S, \text{ i.e. } a (R \cup S) a$$

$$\Rightarrow R \cup S \text{ is reflexive. } \square$$

CONT...

6.27 CONT...

(d) If R and S are symmetric, then $R \cup S$ is symmetric.

Proof. Since R is symmetric $aRb \Rightarrow bRa$

$\Rightarrow \{(a,b),(b,a) \mid a,b \in A\} \subseteq R$

Likewise $\{(c,d),(d,c) \mid c,d \in A\} \subseteq S$

So,

$\{(a,b),(b,a),(c,d),(d,c) \mid a,b,c,d \in A\} \subseteq R \cup S$

$\Rightarrow R \cup S$ is symmetric. $\square$

(e) If R and S are transitive, then $R \cap S$ is transitive

This is false

Counterexample. Let $A = \{a,b,c\}$ and let R and S be relations such that:

$aRb,\ bRc,\ cRa$

and $bSc,\ cSa,\ bSa$

clearly R and S are transitive, but $R \cap S = \{(b,c),(c,a)\}$ is not transitive since $(b,a) \notin R \cap S$. $\square$