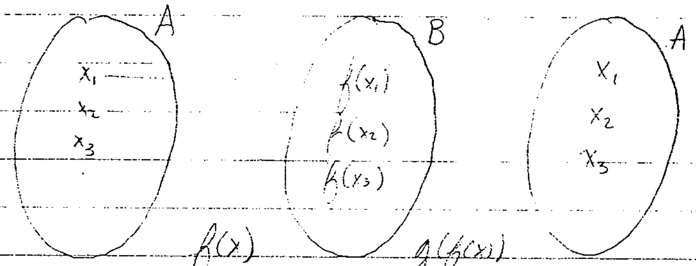
(7.32) Suppose that $f: A \to B$ is any function. Then a function $g: B \to A$ is called:

left inverse for f if $g(f(x)) = x$ for $\forall x \in A$

Right inverse for f if $f(g(y)) = y$ for $\forall y \in B$

a) Prove that f has a left inverse iff f is injective:



Show i) $g(f(x)) = x$ for $\forall x \in A \implies f(x_1) = f(x_2) \implies x_1 = x_2$

$g(f(x)) = x \implies g = i_A \implies f(x_1) = f(x_2) \implies g(f(x_1)) = g(f(x_2)) \implies x_1 = x_2$

$\therefore$ f has a left inverse $\implies$ f is injective ✓

Show ii) $f(x_1) = f(x_2) \implies x_1 = x_2 \implies g(f(x)) = x \ \forall x \in A$

~~If~~ $f(x_1) = f(x_2) \implies x_1 = x_2$ then for $\forall f(x)$ (i.e. elements in the rng f)

$\exists! x \in A$ st $f^{-1}(f(x)) = x$.

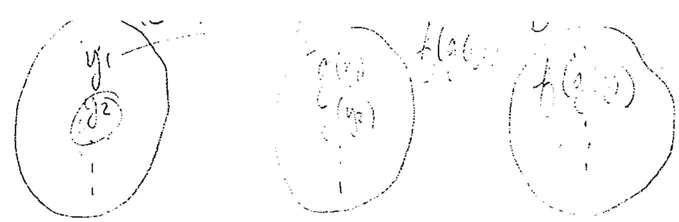
Set $a \in A$

let $g = f^{-1}: B \to A$ st $f^{-1}(f(x)) = g(f(x))$ and define $g(b) = a$ if $b \notin f(A)$

And since all $x \in A$ map to an $f(x) \in B \implies g(f(x)) = x \quad \forall x \in A$ ✓

$\therefore$ f is injective $\implies$ f has a left inverse

$\therefore$ f has left inverse $\iff$ f is injective $\square$



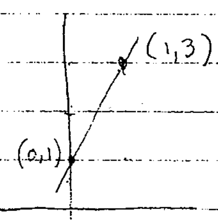
b) Prove that f has a right inverse iff f is surjective.

b) First we will assume $f(g(y)) = y \quad \forall y \in B$.
Recall f is surjective if $\forall y \in B, \exists x \in A$ with $f(x) = y$. Note $g(y) \in A$ so by definition f is surjective.

Now assume f is surjective and show f has a right inverse. Since f is surjective $f^{-1}(\{y\}) \neq \emptyset \quad \forall y \in B$. Note, we can define an equivalence relation on A by $x \sim x'$ if $f(x) = f(x')$. Clearly $x \sim x$ since $f(x) = f(x)$, and if $x \sim x'$ $x' \sim x$ since $f(x) = f(x')$, and if $x \sim x'$ and $x' \sim x''$ then $f(x) = f(x')$ and $f(x') = f(x'') \Rightarrow f(x) = f(x'')$ and $x \sim x''$. Thus $\sim$ is an equivalence relation. For each $y \in B$, choose $x \in f^{-1}(\{y\})$ to represent the equivalence class of x, of the above equivalence relation. Now define $g: B \to A$ by $g(y) = x$, for x the representative of the equivalence class. But $f(g(y)) = f(x) = y$ so g is a left inverse of f.

(8.3) Show that the following pairs of sets are equinumerous by finding a specific bijection between the sets in each pair.

a) $S = [0,1]$ and $T = [1,3]$:



let $f: S \to T$ be defined by $y = 2x+1$ for $x \in S$ and $y \in T$

c) $S = [0,1)$ and $T = (0,1)$, Show S & T equinumerous.
Let $f: S \to T$, defined by $f(x) = \begin{cases} 1/2 & x = 0 \\ 1/(n+1) & \text{if } x = \frac{1}{n}, n > 1 \\ x & \text{if } x \neq 0 \text{ or } x \neq \frac{1}{n}, n > 1 \end{cases}$

f is 1-1 and onto and hence S and T are equinumerous.

d) $S = (0,1)$ and $T = (0, \infty)$: $\lim_{x \to 0} \frac{1}{x} \to \infty$ $\quad \frac{1}{x} - 1 = 0 \Rightarrow x = 1$

define $f: S \to T$, $x \in S$, $y \in T$ by $y = \dfrac{1}{(1/x - 1)}$

This works because $0, 1 \notin S$ and y can never $= 0$ but does grow infinitely.

$\therefore$ Because $\exists$ a bijection between S and T, they are equinumerous

8.13 Prove if $|S| \leq |T|$ and $|T| \leq |S|$ then $|S| = |T|$.
First assume $T \subseteq S$ and their is an injective function $f: S \to T$. Both assumptions imply $|S| \geq |T|$ and $|S| \leq |T|$. We need to construct a bijection from S to T. Since $T \subseteq S$ we can extend the codomain of f to S and view f as a function from S to S.
Define $B = \bigcup_{n=0}^{\infty} f^n(S \setminus T)$, where $f^n(x) = f \circ f^{n-1}(x)$ is defined recursively. Since f is 1-1 $f^n(S \setminus T) \neq f^m(S \setminus T)$ for any $n \neq m$. Also note $S \setminus B \subseteq S(S \setminus T) = T$ since $S \setminus T \subseteq B$.
Define $g: S \to T$ as follows: $g(x) = \begin{cases} f(x) & \text{if } x \in B \\ x & \text{if } x \in S \setminus B (\in T) \end{cases}$
As f was 1-1 from S to T, f will be 1-1 and onto from B to $B \cap T$, since by construction, if $y \in B \cap T$, there was an $x \in B$ with $f(x) = y$. The identity on $S \setminus B$ is a bijection. Hence, g is a bijection from S to T. If $T \not\subseteq S$, w/ $|T| \leq |S|$ Consider the injection $h: T \to S$. h is a bijection with $h(T) \subseteq S$. Use the above argument on $h(T)$ to get a bijection of S with $h(T)$. Then compose with h^{-1} to get the bijection of S with T.

(8.18) Suppose that we let U denote the "Set of all things". Then for any set S we have $S \subseteq U$. In particular, $\mathcal{P}(U) \subseteq U$. Use theorems 8.15 and 8.18 to obtain a contridiction.

Assume Thm. 8.18: for any set S, $|S| < |\mathcal{P}(S)|$
Assume Thm 8.15 If $S \subseteq T$, then $|S| \leq |T|$

If U is the set of all things $\Rightarrow$ any set S must be contained within $U \Rightarrow S \subseteq U$.

Since $\mathcal{P}(U)$ is a set $\Rightarrow$ it too must be contained in U.
$\Rightarrow \mathcal{P}(U) \subseteq U$

And if $\mathcal{P}(U) \subseteq U \Rightarrow |\mathcal{P}(U)| \leq |U|$ (Thm 8.15)

But by (Thm 8.18) $|U| < |\mathcal{P}(U)|$ $\Rightarrow \Leftarrow$ $\square$

(8.20) Is it possible for $\mathcal{P}(S) = \emptyset$ for some set S?

No, the empty set is a subset of S, $\mathcal{P}(S)$ is the <u>set</u> of all <u>subsets</u> of S. So even if $S = \emptyset \Rightarrow \mathcal{P}(S) = \{\emptyset\}$

(8.22) Let A and B be sets. Prove that $\mathcal{P}(A) \cup \mathcal{P}(B) \subseteq \mathcal{P}(A \cup B)$ and show by counter example that equality does not hold.

Let X be a set. If $X \in \mathcal{P}(A) \cup \mathcal{P}(B) \Rightarrow X \in \mathcal{P}(A)$ or $X \in \mathcal{P}(B)$

$$\left.\begin{array}{l} \text{i) If } X \in \mathcal{P}(A) \Rightarrow X \subseteq A \Rightarrow X \subseteq A \cup B \\ \text{ii) If } X \in \mathcal{P}(B) \Rightarrow X \subseteq B \Rightarrow X \subseteq A \cup B \end{array}\right\}$$

$\therefore X \in \mathcal{P}(A)$ or $X \in \mathcal{P}(B) \Rightarrow X \subseteq A \cup B \Rightarrow X \in \mathcal{P}(A \cup B)$

$\Rightarrow \mathcal{P}(A) \cup \mathcal{P}(B) \subseteq \mathcal{P}(A \cup B)$ $\square$ ✓

Show by counter example that $\mathcal{P}(A \cup B) \subsetneq \mathcal{P}(A) \cup \mathcal{P}(B)$

This might be the case if A and B have an intersection

let $A = \{a, b\}$, $B = \{b, c\}$ $\Longrightarrow$ $A \cup B = \{a, b, c\}$

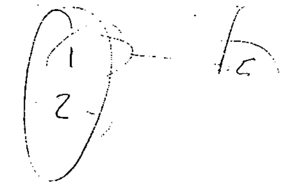
$\Longrightarrow$ $\mathcal{P}(A) = \{\emptyset, \{a\}, \{b\}, \{a,b\}\}$, $\mathcal{P}(B) = \{\emptyset, \{b\}, \{c\}, \{b,c\}\}$

i) $\mathcal{P}(A) \cup \mathcal{P}(B) = \{\emptyset, \{a\}, \{b\}, \{a,b\}, \{c\}, \{b,c\}\}$

ii) $\mathcal{P}(A \cup B) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a,b\}, \{a,c\}, \{b,c\}, \{a,b,c\}\}$

$\mathcal{P}(A \cup B) \not\subseteq \mathcal{P}(A) \cup \mathcal{P}(B)$ because $\{a,b,c\} \in \mathcal{P}(A \cup B)$ but $\{a,b,c\} \notin \mathcal{P}(A) \cup \mathcal{P}(B)$ $\square$

✓



Home Work 2

(7.4) Find all possible functions $f: A \to B$ in each case. Describe the functions by listing their ordered pairs

b) $A = \{4\}$, $B = \{5,6\}$: $\boxed{f = \{(4,5),(4,6)\}}$

$f(a) = a+1$ or $f(a) = a+2$
$= (4,5)$ $= (4,6)$

? these would count if the function were not "into" ?

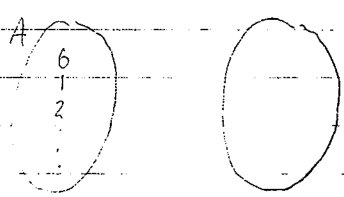
c) $A = \{1,2\}$ and $B = \{5,6\}$

~~f_1 : (1,5)~~	$f_5 = \{(1,5),(2,5)\}$
~~f_2 : (1,6)~~	$f_6 = \{(1,6),(2,6)\}$
~~f_3 : (2,5)~~	$f_7 = \{(1,5),(2,6)\}$
~~f_4 : (2,6)~~	$f_8 = \{(1,6),(2,5)\}$

(7.6) Let $A \subseteq \mathbb{R}$ and define $f: A \to \mathbb{R}$ as given. In each case describe a set A so that f is injective on A. Make A as large as possible.

a) $f(x) = (x+3)^2 - 5$:

$\boxed{\text{let } A = \{x \in \mathbb{R} \mid x \geq -3\}}$



$2x - 1 > 0$
$x > \frac{1}{2}$

✓ b) $f(x) = |2x-1|$:

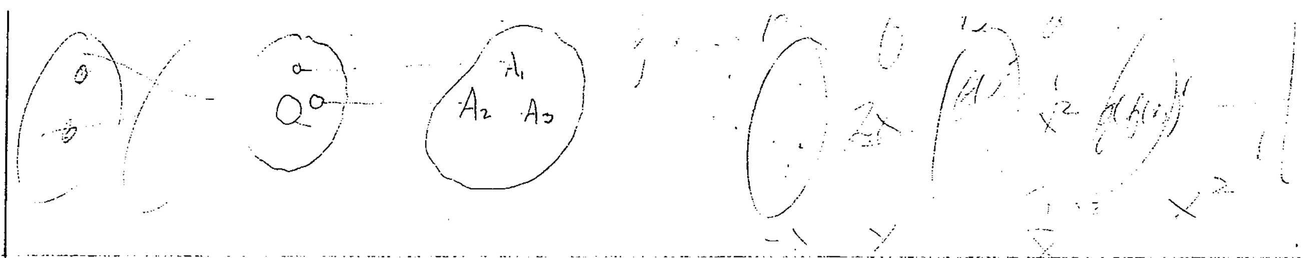
$|2(-1)-1| = |-3| = 3$
$|2(0)-1| = |-1| = 1$
$|2(1)-1| = |1| = 1$

$\boxed{\text{let } A = \{x \in \mathbb{R} \mid x \geq 1/2\}}$

(7.7) Classify each function as injective, surjective, bijective, or none of the above:

b) $f: \mathbb{Z} \to \mathbb{Z}$ defined by $f(n) = n-5$: $\boxed{\text{bijective}}$

g) $f: \mathbb{N} \to \mathbb{Q}$ defined by $f(n) = 1/n$: $\boxed{\text{Injective but not surjective}}$



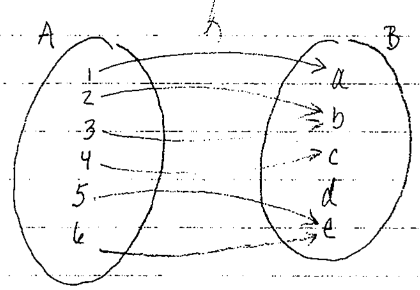
(7.8) a) Let S be the set of all circles in the plane.
Define $f: S \to [0, \infty)$ by $f(C) =$ area of C for $\forall C \in S$.
Is f injective, surjective?

- It is not injective: Circles at different positions can have the same area
- It is surjective: All #'s ≥ 0 will correspond to an area of a circle

b) Let T be the set of all circles in the plane that are centered at the origin. Define $g: T \to [0, \infty)$ by $f(c) =$ area of c for $\forall c \in T$.
Is g injective, surjective?

- Since position is no longer relevant, g is injective and also surjective, i.e. bijective

(7.13) Consider the illustrated function:



a) find $f(S)$, where $S = \{2, 3, 4, 5\}$

- $\boxed{f(S) = \{b, c, e\}}$

b) find $f^{-1}(T)$, where $T = \{a, b, d\}$

$f^{-1}(T)$ if we mean the pre-image, is $\{1, 2, 3\}$

(7.14) Define $f: \mathbb{R} \to \mathbb{R}$ by $f(x) = x^2$. find $f^{-1}(T)$ for each of the following: $x \in \mathbb{R}$ st $f(x) \in T \Leftrightarrow$

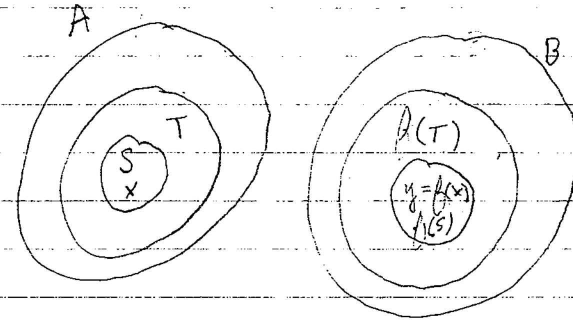
a) $T = \{9\}$ ∴ $f^{-1}(T) = \{-3, 3\}$

b) $T = [4, 9)$: $\boxed{f^{-1}(T) = (-3, -2] \cup [2, 3)}$

c) $T = [-4, 9]$: The only $f(x) \in T$ are from $[0, 9]$, so $\boxed{f^{-1}(T) = [-3, 3]}$

(7.19) Suppose $f : A \to B$ and S and T are subsets of A. Prove or give a counter example:

a) If $S \subseteq T, \Rightarrow f(S) \subseteq f(T)$:



∘ let $y \in f(S) \Rightarrow \exists\, x \in S$ st $y = f(x)$

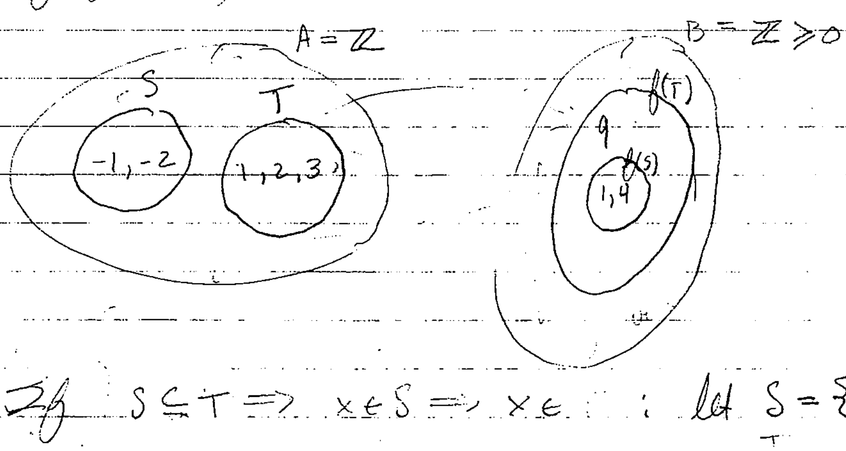
$S \subseteq T \Rightarrow x \in S \Rightarrow x \in T$

Since $x \in T$ & $y = f(x)$

$\Rightarrow y \in f(T)$

∴ $y \in f(S) \Rightarrow y \in f(T) \Rightarrow f(S) \subseteq f(T)$ □

b) If $f(S) \subseteq f(T), \Rightarrow S \subseteq T$:



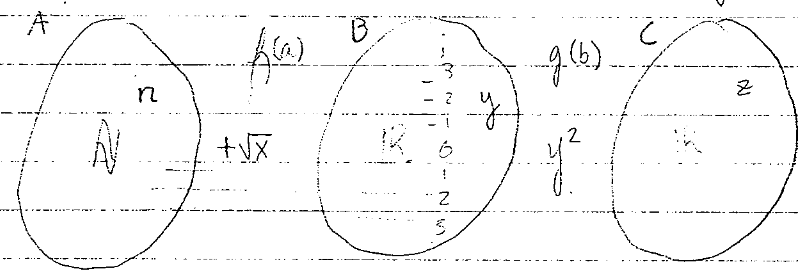
If $S \subseteq T \Rightarrow x \in S \Rightarrow x \in$: let $S = \{-1, -2\}$, $T = \{1, 2, 3\}$

for $x \in A$ define $f : A \to B$ by $f(x) = x^2$

$\Rightarrow f(S) = \{1, 4\}$ and $f(T) = \{1, 4, 9\}$

∴ $f(S) \subseteq f(T)$ however $S \not\subseteq T$. So $f(S) \subseteq f(T) \not\Rightarrow S \subseteq T$ □

(7.26) Find an example of functions $f: A \to B$ and $g: B \to C$ st f and $g \circ f$ are both injective but g is not injective



$g(f(a))$

let $g: B \to C$ be y^2, let $f: A \to B$ be $\overset{\text{positive}}{+}\sqrt{n}$; $g(f(n)) = (+\sqrt{n})^2 = n$

Also, let $A = \mathbb{N}$, let $B = \mathbb{R}$, let $C = \mathbb{R}$

$f: A \to B$ is injective because for $n \in \mathbb{N}$, $n_1 \neq n_2 \Rightarrow +\sqrt{n_1} \neq +\sqrt{n_2}$

$g: B \to C$ is not injective because for $y \in \mathbb{R}$ $-3 \neq 3$ but $(-3)^2 = (3)^2$

$g \circ f: A \to C$ is injective because $+\sqrt{n_1} \neq +\sqrt{n_2} \Rightarrow (+\sqrt{n_1})^2 \neq (+\sqrt{n_2})^2$
$\Rightarrow n_1 \neq n_2$ □

7.27 Find an example of functions $f: A \to B$, $g: B \to C$ such that g & $g \circ f$ onto but f is not.

Let $f: \mathbb{Z} \to \mathbb{Z}$ be defined by $f(n) = 2n$
Let $g: \mathbb{Z} \to \mathbb{Z}$ be defined by $g(n) = \begin{cases} \frac{n}{2} & n \text{ even} \\ \frac{n-1}{2} & n \text{ odd} \end{cases}$

$g \circ f(n) = n$ so $g \circ f$ is surjective
And by definition g is surjective
but f is not since the odds are not in the image of f.