12.3 (bhL), 12.4 (bhL), 12.6, 12.8, 12.12

12.3 (b) $\{\pi, 3\}$ supremum $= \pi$ ✓ (Least Upper Bound)
maximum $= \pi$ ✓

(h) $\{(-1)^n(1+\frac{1}{n}) : n \in \mathbb{N}\}$ supremum $= \frac{3}{2}$ ✓ $(> \frac{5}{4})$
$\{-2, +\frac{3}{2}, -\frac{4}{3}, +\frac{5}{4}\}$ maximum $= \frac{3}{2}$ ✓

(L) $\bigcup_{n=1}^{\infty} [\frac{1}{n}, 2-\frac{1}{n}]$ Supremum $= 2$ ✓ (when $n=\infty$)
maximum = None ✓ because
$\{\cancel{[1,1]}\ \{1\}, [\frac{1}{2}, \frac{3}{2}], [\frac{1}{3}, \frac{5}{3}], \ldots\}$ there are ∞ intervals.

12.4. (b) $\{\pi, 3\}$ infimum $= 3$ (Greatest Lower Bound)
minimum $= 3$.

(h) $\{(-1)^n(1+\frac{1}{n}) : n \in \mathbb{N}\}$ infimum $= -2$
minimum $= -2$

(L) $\bigcup_{n=1}^{\infty} [\frac{1}{n}, 2-\frac{1}{n}]$ infimum $= 0$ (when $n=\infty$)
minimum = None because there are ∞ solutions ($\frac{1}{n}$) going to $\emptyset$

12.6. (a) Let S = nonempty + subset of $\mathbb{R}$. Prove $\sup(S)$ unique
Suppose m and n are both $\sup(S)$. Both m and n are the upper bounds of S. By definition of supremum, m is the Least upper bound so $m \leq n$. Also, n is the Least upper bound so $n \leq m$. Therefore, $m = n$ and the $\sup(S)$ is unique!

(b) Suppose m and n are both maxima of S.
Prove $m = n$.
Since m is a ~~maximum~~ of S, then $m \in S$ where $m \geq s\ \forall s \in S$. However, $n \in S$ so $m \geq n$.
Since n is a maximum of S, then $n \in S$ where $n \geq s\ \forall s \in S$. Since $m \in S$, then $n \geq m$. Therefore, $m = n$

12.8. Let S and T be nonempty, bounded subsets of $\mathbb{R}$ $S \subseteq T$. Prove $\inf(T) \le \inf(S) \le \sup(S) \le \sup(T)$

The $\inf(T) \le t \;\forall t \in T$. Since $S \subseteq T$ then $\inf(T) \le s$ $\forall s \in S$. Since $\inf(T)$ is ~~the~~ lower bound for S then $\inf(T) \le \inf(S)$ which is the greatest lower bound for S. Then, Let $s_0 \in S$ and $\inf(S) \le s \;\forall s \in S$. So, $\inf(S) \le s_0$. Also, $\sup(S) \ge s \;\forall s \in S$ so, $\sup(S) \ge s_0$. Therefore, $\inf(S) \le s_0 \le \sup(S)$. Finally, the $\sup(T) \ge t \;\forall t \in T$. Since $S \subseteq T$, then $\sup(T) \ge s$ $\forall s \in S$. Since $\sup(T)$ is ~~the~~ upper bound for S then $\sup(T) \ge \sup(S)$ which is the least upper bound for S. Therefore, $\inf(T) \le \inf(S) \le \sup(S) \le \sup(T)$.

12.12 D = nonempty set

$f: D \to \mathbb{R}$ $\quad g: D \to \mathbb{R}$ $\quad f+g: D \to \mathbb{R}$

$(f+g)(x) = f(x) + g(x)$

(a) $f(D)$ and $g(D)$ bounded above

Let $m = \sup(f(D))$ and $n = \sup(g(D))$. Then $\forall x \in D$ $(f+g)(x) = f(x) + g(x) \le (m+n)$. $(m+n)$ is an upper bound for $(f+g)(D)$. The least upper bound is $\sup((f+g)(D))$, which is $\le$ the upper bound. $(\sup[(f+g)(D)] \le (m+n))$

Therefore, $\sup[(f+g)(D)] \le \sup(f(D)) + \sup(g(D))$

(b) Let $D = [0,1]$, $f(x) = x$, $g(x) = 1-x$ $\quad f(D) = [0,1]$

$\sup(f(D)) = 1 \quad \sup(g(D)) = 1 \quad g(D) = [0,1]$

$(f+g)(D) = x + 1 - x = 1$

$\sup[(f+g)(D)] = 1 < 2 = \sup(f(D)) + \sup(g(D))$

(c) $f(D)$ and $g(D)$ ~~bounded below~~. $\inf[(f+g)(D)] \ge \inf(f(D)) + \inf(g(D))$

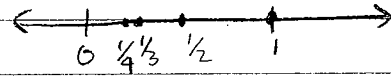
Let $m = \inf(f(D))$ and $n = \inf(g(D))$ Then $\forall x \in D$ $(f+g)(x) = f(x) + g(x) \ge (m+n)$. $(m+n)$ is a lower bound for $(f+g)(D)$. The greatest lower bound is $\inf[(f+g)(D)]$, which is $\ge$ the lower bound.

So, $\inf[(f+g)(D)] \ge (m+n)$ which is

$\inf[(f+g)(D)] \ge \inf(f(D)) + \inf(g(D))$

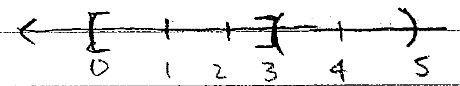
13.3(ab), 13.4(ab), 13.5(cd), 13.7(abf), 13.12, 13.20(ac), 13.21(bd)

13.3 (a) $\{\frac{1}{n} : n \in \mathbb{N}\}$



These are individual points so $\text{int}(S) = \{\ \}$

(b) $[0,3] \cup (3,5)$



All points of S included in $\text{int}(S) = (0,5)$

13.4 (a) The boundary points are $\{0\} \cup \{\frac{1}{n} : n \in \mathbb{N}\}$ because 0 is the edge of S^c and $\{\frac{1}{n} : n \in \mathbb{N}\}$ defines points on the boundary of S.

(b) The boundary points are $\{0, 5\}$ because they are on the edge of S.

13.5 c) $\text{bd}(\mathbb{Q}) = \mathbb{R}$ since $N(r;\varepsilon)$ contains a rational # for all $r \in \mathbb{R}$ by the Theorem that says given 2 real numbers $x<y$, $\exists\, r \in \mathbb{Q}$ with $x<r<y$. Since $\text{bd}\,\mathbb{Q} \not\subseteq \mathbb{Q}$, $\mathbb{Q}$ is not closed. Since $\text{bd}\,\mathbb{Q} \not\subseteq \mathbb{Q}^c$, $\mathbb{Q}$ is not open.

d) $\bigcap_{n=1}^{\infty} (0, 1/n) = \emptyset$ which is closed and open

13.7 S and T subsets of $\mathbb{R}$

(a) If P is all isolated pts. in S, then P is a closed set.

Let $S = \{\frac{1}{n} : n \in \mathbb{N}\}$ $P = S$

S is not a closed set because a boundary point of S is $\{0\}$. However, $\text{bd}(S) = \{0\} \cup S \not\subseteq S$. $\therefore$ S is not closed so P is not closed.

(b) If S is closed, then $cl(int S) = S$.
Let $S = [0,1] \cup \{2\}$ the $int(S) = (0,1)$
the $cl(int(S))$ is all the accumulation points in $int(S)$. $\therefore$ $cl(int(S)) = [0,1] \neq S$ because it does not include the point $\{2\}$.

(f) $bd(S \cup T) = bd(S) \cup bd(T)$
Let $S = [0,2]$ and $T = [1,3]$
$bd(S \cup T) = \{0,3\}$, $bd(S) = \{0,2\}$, and $bd(T) = \{1,3\}$ so $bd(S) \cup bd(T) = \{0,1,2,3\}$
$\neq bd(S \cup T) = \{0,3\}$.

13.12 show $N^*(x;\varepsilon)$ is open
Let $y \in N^*(x;\varepsilon)$. Let $\delta = \min\{x+\varepsilon-y, |x-y|, |x-\varepsilon-y|\}$
We need to show $N(y;\delta) \subseteq N^*(x;\varepsilon)$
(case 1) If $y > x$ then for any $z \in N(y;\delta)$

$$x-y = -|x-y| \leq -\delta < z-y < \delta \leq x+\varepsilon-y$$

implies $x < z < x+\varepsilon$ which implies $z \in N^*(x;\varepsilon)$

(case 2) If $y < x$ then for any $z \in N(y;\delta)$

$$(x-\varepsilon)-y = -|(x-\varepsilon)-y| \leq -\delta < z-y < \delta \leq |x-y| = x-y$$

implies $x-\varepsilon < z < x$ which implies $z \in N^*(x;\varepsilon)$

In both cases, we see that every point of $N^*(x;\varepsilon)$ is an interior point.

13.20 Let S and T be subsets of $\mathbb{R}$

(a) $cl(cl(S)) = cl(S)$

Theorem 13.17 (b) states $cl(S)$ is a closed set. Since $cl(S)$ is closed, $cl(S) = cl(cl(S))$ by Theorem 13.17 (c).

13.20 c) Show $cl(S \cap T) \subseteq (cl(S)) \cap (cl(T))$

Recall $cl(S\cap T) = (S \cap T) \cup (S \cap T)'$

If $x \in S \cap T$ then $x \in S$ and $x \in T$

which implies $x \in S \cup S'$ and $x \in T \cup T'$

or $x \in cl(S) \cap cl(T)$

Suppose $x \in (S\cap T)'$, then $\forall \varepsilon$ $N^*(x,\varepsilon) \cap (S \cap T) \neq \emptyset$.

Since $S \cap T \subseteq S$ and $S \cap T \subseteq T$, then

$N^*(x;\varepsilon) \cap (S \cap T) \subseteq N^*(x;\varepsilon) \cap S \neq \emptyset$

and $N^*(x;\varepsilon) \cap (S\cap T) \subseteq N^*(x;\varepsilon) \cap T \neq \emptyset$.

Thus $x \in cl(S)$ and $x \in cl(T)$.

Hence, $cl(S \cap T) \subseteq cl(S) \cap cl(T)$

13.21 b) Show $int(int S) = int S$.

If we show $int S$ is open, then $int(int S) = int S$ by 13.7. Note, $x \in int S$ if there exists $\varepsilon > 0$ such that $N(x;\varepsilon) \subseteq S$. For an arbitrary $x \in int S$ we need to show $N(x;\varepsilon) \subseteq int S$.

To do this we will show $\forall y \in N(x;\varepsilon)$, $y \in int S$.

$y \in int S$ if $\exists \delta$ such that $N(y;\delta) \subseteq S$.

Let $\delta = \min\{x+\varepsilon-y,\ y-(x-\varepsilon)\}$. For any $z \in N(y;\delta)$

$$x-\varepsilon-y = -(y-(x-\varepsilon)) \leq -\delta < z-y < \delta \leq x+\varepsilon-y$$

which implies $x-\varepsilon < z < x+\varepsilon$. Hence $z \in N(x;\varepsilon) \subseteq S$.

Hence, $y \in int S$, $\forall y \in N(x;\varepsilon)$. Hence, $N(x;\varepsilon) \subseteq int S$.

This was true for arbitrary $x \in int S$. Hence, $bd(int S) \subseteq int S$. Thus $int S$ is open.

(d) $\text{int}(S) \cup \text{int}(T) \subseteq \text{int}(S \cup T)$

Let $x \in \text{int}(S) \cup \text{int}(T)$ Then $x \in \text{int}(S)$ or $x \in \text{int}(T)$. For every neighborhood of x such that $N(x;\varepsilon) \subseteq S \Rightarrow N(x;\varepsilon) \subseteq (S \cup T)$. Similarly $N(x;\varepsilon) \subseteq T \Rightarrow N(x;\varepsilon) \subseteq S \cup T$

So, $x \in \text{int}(S \cup T)$ and $\text{int}(S) \cup \text{int}(T) \subseteq \text{int}(S \cup T)$