ALGEBRAIC TOPOLOGY – HOMEWORK

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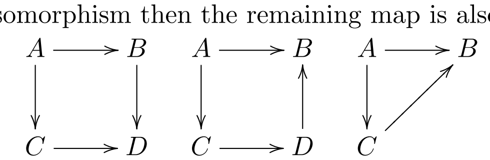
(1) Construct an explicit deformation retract of the torus with one point deleted to $S^1 \vee S^1$. (Hint: Think about the torus as a quotient space of a square.)
(2) Construct an explicit deformation retract of $\mathbb{R}^n \setminus \{0\}$ onto S^{n-1}.
(3) Show that homotopy equivalence is an equivalence relation.
(4) Given $v, e, f \in \mathbb{Z}$ satisfying $v - e + f = 2$. Construct a cell complex for S^2 with v 0-cells, e 1-cells and f 2-cells.
(5) Show $S^1 * S^1 \cong S^3$.
(6) Show that the space obtained from S^2 by attaching n 2-cells along any collection of circles in S^2 is homotopy equivalent to the wedge sum of $n+1$ 2-spheres.
(7) Show that a CW-complex is contractible if it is the untion of 2 contractible CW-complexes whose intersection is also contractible.
(8) If X is a set of k points for some positive integer k, determine $H_i(X)$ for all $i \geq 0$.
(9) Suppose $\{f_i : I \to X\}_{i=0}^n$ are a collection of paths on X such that $f_{i-1}(1) = f_i(0)$ for $1 \leq n$. Show that $\sum_{i=0}^n f_i$ is homologous to $f_0 * f_1 * \cdots * f_n$.
(10) If $X = S^1$ determine $H_i(X)$ for $i = 0, 1$
(11) If $X = S^1 \vee S^1$ determine $H_i(X)$ for $i = 0, 1$.
(12) Let A be an arccomponent of X and $f : A \to X$ be the inclusion map. Prove that $f_* : H_n(A) \to H_n(X)$ is a monomorphism. Indeed if X is the disjoint union of arccomponents $\{X_i\}_{i \in I}$ then $H_n(X) = \bigoplus_{i \in I} H_n(X_i)$.
(13) Let A be retract of X with retract map $r : X \to A$. Let $i : A \to X$ be the inclusion mapping. Prove $r_* : H_n(X) \to H_n(A)$ is an epimorphism and $i_* : H_n(A) \to H_n(X)$ is a monomorphism. Then show that $H_n(X)$ is isomorphic to $\text{im}(i_*) \oplus \text{ker}(r_*)$.
(14) Let X be arcwise connected. If $f : X \to X$ is any continuous map then $f_* : H_0(X) \to H_0(X)$ is the identity.
(15) Let $f : X \to Y$ be a continuous map such that $f(x_0) = y_0$ and $f_\# : \pi_1(X, x_0) \to \pi_1(Y, y_0)$ be the induced map on the fundamental groups. If ϕ_X and ϕ_Y are the Hurewicz maps from the fundamental groups to the first homology groups, prove $\phi_Y \circ f_\# = f_* \circ \phi_X$.
(16) Prove that $\Delta_n(X, A)$ is a free abelian group generated by the cosets of the singular n-simplices not contained in A.
(17) Prove the following:
   (a) $H_n(A) \cong H_n(X)$ if and only if $H_n(X, A) = 0$ for all $n \geq 0$.
   (b) $H_n(X, A) \cong H_n(X)$ if and only if $H_n(A) = 0$ for all $n \geq 0$.
   (c) $H_n(X, A) = 0$ for $n \leq m$ if and only if $H_n(A) \cong H_n(X)$ for $n \leq m-1$ and $i_* : H_m(A) \to H_m(X)$ is onto.
(18) If $A = \emptyset$ what is $H_n(X, A)$ for all $n \geq 0$? Justify your answer.
(19) If $A \neq \emptyset$ and A is acyclic show that $H_n(X, A) \cong H_n(X)$ for $n \geq 1$.
(20) Prove for $n \neq m$, D^n and D^m are not homeomorphic.
(21) Prove any homeomorphism of D^n onto itself maps S^{n-1} onto S^{n-1}.
(22) Prove that any map $f : S^n \to S^n$ which has nonzero degree must be onto.
(23) If $f : S^n \to S^n$ be a map without fixed points, show that $\deg(f) = (-1)^{n+1}$.
(24) For n even, show that any map $f : \mathbb{RP}^n \to \mathbb{RP}^n$ has a fixed point.
(25) In each of the following commutative diagrams, prove that if all of the maps but one is an isomorphism then the remaining map is also an isomorphism.



(26) Given an example of a commutative diagram so the middle vertical map is nonzero but all the remaining vertical maps are the zero map:

$$\begin{array}{ccccccccc} A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & D & \longrightarrow & E \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ A' & \longrightarrow & B' & \longrightarrow & C' & \longrightarrow & D' & \longrightarrow & E' \end{array}$$

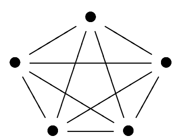
(27) Carefully write up the rest of the proof of the Braid Lemma using the notation introduced in class. (I would like to see exactness at G_1 and G_3.)

(28) Let $X_k = S^1 \bigcup_{\phi_k} D^2$, where $\phi_k(z) = z^k$ for all $z \in S^1$. Compute $H_*(X_k)$.

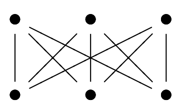
(29) Find the homology groups with integer coefficients of the 2-sphere union with the segment connecting the north pole and the south pole. (Hint: The CW complex should have 2 0=cells, 3 1-cells and 2 2-cells. Think about how they are attached.)

(30) Find the homology groups with integer coefficients of $S^2 \vee S^1$. Compare with the groups with the groups you obtained in the previous problem.

(31) Find the homology groups with integer coefficients of K_5 the complete graph on 5 vertices. (Hint: the CW complex has 5 0-cells and 10 1-cells.)



(32) Find the homology groups with integer coefficients of $K_{3,3}$ the complete bipartate graph where the vertices are partitioned in two disjoint subsets of 3 points each.



(33) Find the homology groups with integer coefficients of the space obtained by attaching two 2-cells to a circle one via a degree 2 map and the other via a degree 3 map.

(34) For a CW-complex show that $\sum_\tau [\omega : \tau][\tau : \sigma] = 0$ for all $n+1$-cells σ and $n-1$-cells ω and τ ranges over all n-cells.

(35) Compute $H_i(S^n \times S^m)$ for all i.

(36) Compute $H_i(\mathbb{RP}^2 \times \mathbb{RP}^2)$ for all i.

(37) If X and Y are finite CW-complexes, show that $\chi(X \times Y) = \chi(X)\chi(Y)$.

(38) Let U and V be open sets such that $U \cap V$ is contractible. Express the homology of $H_i(U \cup V)$ in terms of $H_i(U)$ and $H_i(V)$ for all $i \in \mathbb{Z}$.

(39) Use Mayer-Vietoris to compute the homologies of the union of three discs with a common boundary.

(40) Let $A, B \subseteq S^n$ with A and B disjoint such that A is homeomorphic to S^m and B is homeomorphic to S^k with $0 \leq m, k < n$. Determine the homologies of $S^n - (A \cup B)$.

(41) Let $A, B \subseteq S^n$ with $A \cap B$ a single point and A is homeomorphic to S^m and B is homeomorphic to S^k with $0 \leq m, k < n$. Determine the homologies of $S^n - (A \cup B)$.

(42) Let $f : \mathbb{RP}^n \to \mathbb{RP}^m$ with $n > m > 0$. Show $f_\# : \pi_1(\mathbb{RP}^n) \to \pi_1(\mathbb{RP}^m)$ is trivial.

(43) Show that $\mathbb{RP}^2$ is not a retract of $\mathbb{RP}^3$.

(44) On the unit circle S^1 in the plane, let $\theta = \arctan(y/x)$. Show that $d\theta$ is a closed 1-form which is not exact.

(45) For $\omega \in \Omega^1(M)$, verify $d\omega(X, Y) = X(\omega(Y)) - Y(\omega(X)) - \omega([X, Y])$.

(46) Find $H^i_\Omega(\mathbb{R}^n; \mathbb{R})$ for all i.

(47) Find $H^i_\Omega(S^1; \mathbb{R})$ for $i = 0, 1$.

(48) Describe Green's Theorem and the Divergence Theorem in terms of forms.

(49) Give examples of at least three groups G such that
   (a) $\mathrm{Ext}(\mathbb{Z}_n, G) = 0$.
   (b) $\mathrm{Tor}(\mathbb{Z}_n, G) = 0$.

(50) Compute $H^i(\mathbb{RP}^n)$ for all $i \in \mathbb{Z}$.

(51) Compute $H^i(\mathbb{RP}^n; \mathbb{Z}_2)$ for all $i \in \mathbb{Z}$.

(52) If X arises from attaching a 2-disk to a circle by a map of circles of degree 4. Compute the integral homology groups of X with coefficient groups $\mathbb{Z}_2$ and $\mathbb{Z}_4$ respectively.
(53) Show $H^1(X)$ is torsionfree for all X.
(54) Triangulate the torus and use simplicial theory to compute its homology.
(55) Consider the triangulation of S^2 given by an octahedron. This is invariant under the antipodal map and gives a CW-decomposition of $\mathbb{RP}^2$ into 4 triangles. Show that this is not a triangulation of $\mathbb{RP}^2$.
(56) Let K be any subdivision of Δ_n and let $g : K \to \Delta_n$ be a simplicial approximation to the identity. Show that the number of simplices of K that map onto Δ_n is odd. Note that g must map $\partial\Delta_n$ into itself.
(57) Let K be a simplicial complex and K' be its barycentric subdivision. Let $\mathcal{Y} : C_*(K) \to C_*(K')$ be the subdivision chain map and let $\phi : K' \to K$ be a simplicial approximation to the identity. Show $\phi_\Delta \circ \mathcal{Y}$ is the identity on $C_*(K)$.