3-0235 — 50 SHEETS — 5 SQUARES
3-0236 — 100 SHEETS — 5 SQUARES
3-0237 — 200 SHEETS — 5 SQUARES
3-0137 — 200 SHEETS — FILLER
COMET

Ex 14.3

a) $[1,3)$

Find open cover that has no finite subcover

$$A_n = \left(0, 3 - \frac{1}{n}\right) \quad \forall n \in \mathbb{N}$$

We have that $\{A_n\}_{n \in \mathbb{N}}$ is an open cover of $[1,3)$ and it has no finite subcover

In fact:

$$[1,3) \subset \bigcup_{n=1}^{\infty} A_n$$

Now let $\mathcal{G} = \{A_{n_1}, \dots, A_{n_k}\}$ be a finite subfamily of $\mathcal{F}$ $(\mathcal{F} = \{A_n : n \in \mathbb{N}\})$ and if $m = \max\{n_1, \dots, n_k\}$ then

$$A_{n_1} \cup \dots \cup A_{n_k} = A_m = \left(0, 3 - \frac{1}{m}\right)$$

But $[1,3) \not\subset A_m$ therefore $\mathcal{G}$ is not an open cover of $[1,3)$ therefore we conclude that A_n has no finite subcover

3-0235 — 50 SHEETS — 5 SQUARES
3-0236 — 100 SHEETS — 5 SQUARES
3-0237 — 200 SHEETS — 5 SQUARES
3-0137 — 200 SHEETS — FILLER
COMET

c) $\{ 1/n : n \in \mathbb{N} \}$

An open cover that has no finite subcover is

$A_n = (\frac{1}{n}, 2) \quad n \in \mathbb{N}$

$\{ \frac{1}{n} : n \in \mathbb{N} \} \subset \bigcup_{n=1}^{\infty} A_n$

As before let's assume $G = \{ A_{n_1}, \dots, A_{n_k} \}$ be a finite subcover of $\mathcal{F}$ ($\mathcal{F} = \{ A_n : n \in \mathbb{N} \}$) and if $m = \max \{ n_1, \dots, n_k \}$ then

$$A_{n_1} \cup \dots \cup A_{n_k} = A_m = (\frac{1}{m}, 2)$$

By the Archimedean theorem there exists $p \in \mathbb{N}$, $p > m$ s.t. $\frac{1}{p} < \frac{1}{m}$

$\therefore \{ \frac{1}{n} : n \in \mathbb{N} \} \not\subset A_m$ therefore we conclude that A_n has no finite subcover

Ex 14.4

The intersection of any collection of compact sets is compact.

Pf

From Heine-Borel theorem: a subset S of $\mathbb{R}$ is compact iff S is closed & bounded

By corollary 13.1 (a) the intersection of any collection of closed sets is closed

3-0235 — 50 SHEETS — 5 SQUARES
3-0236 — 100 SHEETS — 5 SQUARES
3-0237 — 200 SHEETS — 5 SQUARES
3-0137 — 200 SHEETS — FILLER
COMET

Therefore the intersection of any collection of compact sets is closed since to be compact means to be closed and bounded

Let's S_i denote the compact sets

$$S_i \text{ compact} \iff \begin{cases} S_i \text{ closed} \\ S_i \text{ bounded} \end{cases}$$

S_i bounded means that we can always find a ball centered at 0 with radius $r_i > 0$ that completely contains S_i. Therefore it is guaranteed that the ball cover the intersection because any x in the intersection of S_i is also in S_i and it's also in the ball.

Ex 14.5 a)

If S_1 & S_2 are compact subsets of $\mathbb{R}$ then $S_1 \cup S_2$ is compact.

Pf

From Heine-Borel theorem a subset S of $\mathbb{R}$ is compact iff S is closed & bounded

By corollary 13.1 (b) the union of any collection of closed sets is closed

Therefore since S_1 & S_2 are closed because they are compact, also $S_1 \cup S_2$ is closed.

Now S_1 & S_2 are bounded because compact

Let $m = \min\{\inf S_1, \inf S_2\}$

$M = \max\{\sup S_1, \sup S_2\}$

This means that $\boxed{\begin{array}{l} \sup(S_1 \cup S_2) = M \\ \inf(S_1 \cup S_2) = m \end{array}}$

this takes a little more work, but certainly M is an upper bound & m a lower bound which is all you need.

$\therefore$ $S_1 \cup S_2$ is bounded by M and m.

$\therefore$ we have proved that $S_1 \cup S_2$ is closed and bounded that implies $S_1 \cup S_2$ is compact.

Ex 14.8

a) Proof by def. of compactness

suppose S is compact & T is a closed subset of S

Let $\{U_\alpha\}_{\alpha \in A}$ be an open cover of T

T^c is open since T is closed

S is compact so we can find a finite subcover

$\{U_i\}_{i=1}^{n}$ of S

Now

$\{U_i\}_{i=1}^{n} \setminus T^c$ is a finite subcover (of $\{U_\alpha\}$) of T

$\therefore$ T is compact

3-0235 — 50 SHEETS — 5 SQUARES
3-0236 — 100 SHEETS — 5 SQUARES
3-0237 — 200 SHEETS — 5 SQUARES
3-0137 — 200 SHEETS — FILLER
COMET

b) Proof by Heine-Borel Theorem

Suppose S is a compact subset of $\mathbb{R}$ and T is a closed subset of S; $T \subset S$

Since S is compact, S is closed and bounded so T is also bounded

T is also closed from the assumption $\therefore$ by Heine-Borel theorem T is compact.

Ex 14.11

Cantor Set $C = \bigcap_{k=1}^{\infty} A_k$

$$A_1 = \left[0, \tfrac{1}{3}\right] \cup \left[\tfrac{2}{3}, 1\right]$$

$$A_2 = \left[0, \tfrac{1}{9}\right] \cup \left[\tfrac{2}{9}, \tfrac{1}{3}\right] \cup \left[\tfrac{2}{3}, \tfrac{7}{9}\right] \cup \left[\tfrac{8}{9}, 1\right]$$

a) C is compact

C is the complement of a union of open intervals, therefore closed

C is a subset of $[0,1]$ therefore bounded.

By Heine-Borel theorem we conclude that it is compact

b) $x \in C$ iff $a_n \in \{0, 2\}$ $n \in \mathbb{N}$

suppose $x \in [0,1]$, then the ternary expansion of x is denoted by the sequence $(x_1, x_2, \dots)$ if

$x = x_1 3^{-1} + x_2 3^{-2} + x_3 3^{-3} + \dots$ $x_i \in \{0,1,2\}$

If the ternary expansion of x is of the form $(x_1, x_2, \dots x_n, 1, 0000)$ then we can find a

second ternary expansion of x of the form
$(x_1, \dots x_n, 0, 2222..)$ $\circledast$

Now suppose $x \in C = \bigcap A_k$, since $x \in A_1$ $\rightarrow$
$0 \le x \le 1/3$ or $\frac{2}{3} \le x \le 1$, so x_1 is either 0 or 2
(I used $\circledast$) ✓

In the same way since $x \in A_2$, $x_2 = 0$ or $x_2 = 2$
and so on for all A_k

$\therefore$ every digit in the ternary expansion is 0 or 2.

Conversely suppose $x \notin C$ then $x \notin A_k$ for some k
so x is in one of the excluded $1/3$ intervals so
$x_k = 1$,

$\therefore$ $x \in C$ iff $x_k \in \{0,2\}$ $\forall k \in \mathbb{N}$.

c) C is uncountable

$f : C \to [0,1]$
where $f\left(\sum_{k=1}^{\infty} x_k \cdot 3^{-k}\right) = \sum_{k=1}^{\infty} \frac{x_k}{2} \cdot 2^{-k}$ that
means that for $x \in C$, $x_k \in \{0,2\}$ $\forall k \in \mathbb{N}$

All the 2's in the ternary expansion are replaced with 1's so we get a binary expansion of a number in $[0,1]$

f is surjective because every possible binary expansion of a number in $[0,1]$ is represented in the image of f since every sequence of 0 and 2 is the ternary expansion of some $x \in C$

3-0235 — 50 SHEETS — 5 SQUARES
3-0236 — 100 SHEETS — 5 SQUARES
3-0237 — 200 SHEETS — 5 SQUARES
3-0137 — 200 SHEETS — FILLER
COMET

3-0235 -- 50 SHEETS — 5 SQUARES
3-0236 — 100 SHEETS — 5 SQUARES
3-0237 — 200 SHEETS — 5 SQUARES
3-0137 — 200 SHEETS — FILLER

COMET

$\therefore$ C is uncountable because $[0,1]$ is uncountable.

d) C contains no intervals

let $(a,b) \subseteq [0,1]$ and assume $a<b$.

Let $M = \{ n \in \mathbb{N} : -\log_3 (b-a) < n \}$

we know that $b-a<1$ since $(a,b) \subseteq [0,1]$ that implies that $-\log_3 (b-a) > 0$ and $\in \mathbb{R}$

By the Archimedean property, there is some $m \in M$ s.t. $m \leq k$ for all $k \in M$. So:

$-\log_3 (b-a) < m$

$\hookrightarrow 3^{-(-\log_3 (b-a))} > 3^{-m}$

$\hookrightarrow 3^{\log_3 (b-a)} > 3^{-m} \leftrightarrow b-a > 3^{-m} = |b-a| < 3^{-m}$

But A_m is the union of subsets of $[0,1]$ of length 3^{-m} that implies $(b,a) \not\subseteq A_m$

So $(b,a) \not\subseteq C$

e) $\frac{1}{4} \in C$ and not an endpoint

The base 3 expansion of $\frac{1}{4}$ is 0.02020202

$\therefore$ by part b) $\frac{1}{4} \in C$

$x \in A_k$ is an endpoint if $x=0$, $x=1$, $x=3^{-k}$ $k \in \mathbb{N}$

$\therefore \frac{1}{4} \neq 0, 1, 3^{-k}$ and we conclude that $\frac{1}{4}$ is not an endpoint.

3-0235 — 50 SHEETS — 5 SQUARES
3-0236 — 100 SHEETS — 5 SQUARES
3-0237 — 200 SHEETS — 5 SQUARES
3-0137 — 200 SHEETS — FILLER
COMET

Ex 14.9

Uncountable open cover $\mathcal{F}$ of $\mathbb{R}$ s.t. $\mathcal{F}$ has no finite subcover

$$\mathcal{F} = \{ (-n, n), \ n \in \mathbb{N} \}$$

It is uncountable and $\mathcal{F} \supset \mathbb{R}$. Therefore $\mathcal{F}$ has no finite subcover

$\mathcal{F}$ does not contain a countable subcover

3-0235 — 50 SHEETS — 5 SQUARES
3-0236 — 100 SHEETS — 5 SQUARES
3-0237 — 200 SHEETS — 5 SQUARES
3-0137 — 200 SHEETS — FILLER
COMET

Ex 14.12

$S \subset \mathbb{R}$

S compact iff every infinite subset of S has an accumulation point in S

- Assume S compact and consider $A \subset S$, $|A| = \infty$

Now construct an injective sequence x_n so that

$x_{n+1} \in A - \{x_1, x_2 .. x_n\}$

Since S is compact and $x_n \in S$, x_n has a convergent subsequence $x_{k_n} \to x \in S$

So x is an accumulation point of A

Conversely:

let's assume all infinite subsets of have an accumulation point and consider any sequence x_n in S

Let A = range (x_n). If A is finite, x_n takes some values infinitely often ∴ it has a convergent subsequence

If A is ∞ it has an accumulation point by the hypothesis. Then every $N^*(x, 1/n)$ has infinitely many points of A so we can define a sequence k_n s.t. x_{k_n} is in the $N^*(x, 1/n) - \{x_1 \dots x_{k_{n-1}}\}$ and thus subsequence converges to x

↳ Some of the material used for this proof comes from the web!