HW #7 | 10/29/09

3-0235 — 50 SHEETS — 5 SQUARES
3-0236 — 100 SHEETS — 5 SQUARES
3-0237 — 200 SHEETS — 5 SQUARES
3-0137 — 200 SHEETS — FILLER
COMET

Ex 19.3

a) $s_n = (-1)^n = (-1, 1, -1, 1 \dots)$

The set S of subsequential limits is

$S = \{-1, 1\}$

$\lim\inf s_n = -1$

$\lim\sup s_n = 1$

d) $v_n = n \sin \frac{n\pi}{2}$

$= (1, 0, -3, 0, 5, 0, -7, 0, \dots)$

$S = \{\infty, 0, -\infty\}$

$\lim\inf v_n = -\infty$

$\lim\sup v_n = \infty$

Ex 19.4

a) $w_n = \frac{(-1)^n}{n} = (-1, \frac{1}{2}, -\frac{1}{3}, \frac{1}{4}, -\frac{1}{5}, \frac{1}{6}, \dots)$

$S = \{0\}$

$\lim\inf w_n = \lim\sup w_n = 0$

b) $(x_n) = (0, 1, 2, 0, 1, 3, 0, 1, 4, \dots)$

$S = \{0, 1, \infty\}$

$\lim\inf (x_n) = 0$

$\lim\sup (x_n) = \infty$

Ex 19.8

If (s_n) is a subsequence of (t_n) and (t_n) is a subsequence of (s_n) can we conclude that $(s_n) = (t_n)$?

FALSE

Counterexample

$$(s_n) = (-1)^n = (-1, 1, -1, 1, -1, 1 \ldots)$$
$$(t_n) = (-1)^{n+1} = (1, -1, 1, -1, 1, \ldots)$$

$$(s_n) \neq (t_n)$$

Ex 19.9

(s_n) = bounded sequence

$\lim\inf s_n = \lim\sup s_n = S$

By theorem 19.11: if s_n is a bounded sequence and $s = \lim\sup s_n$ then $\forall \varepsilon > 0$ $\exists$ a number N s.t. $n > N \Rightarrow s_n < s + \varepsilon$

Similarly if $s = \lim\inf s_n$ then $\forall \varepsilon > 0$ we have $s_n > s - \varepsilon$

So

$$s - \varepsilon < s_n < s + \varepsilon$$

$\hookrightarrow$ $-\varepsilon < s_n - s < \varepsilon$

$\hookrightarrow$ $|s_n - s| < \varepsilon$ that is the definition of limit

$\therefore$ s_n converges to S

Ex 19.17

If $\limsup s_n = +\infty$, $k>0$

$\Rightarrow \limsup (k s_n) = +\infty$

Proof by theorem 19.14

Let $s = \limsup s_n$ and $t = \limsup (k s_n)$

By corollary 19.12, $\exists$ a subsequence (s_{n_m}) of (s_n) s.t.

$$\lim_{m\to\infty} s_{n_m} = s$$

Now $\lim\limits_{m\to\infty} k = k$ so $\lim\limits_{m\to\infty} k\, s_{n_m} = ks$

Thus $ks \le \limsup k s_n = t$

Similarly let $(k s_{n_m})$ be a subsequence of $(k s_n)$ that converges to t. Then since $k>0$

$$\lim_{m\to\infty} s_{n_m} = \lim_{m\to\infty} \frac{k s_{n_m}}{k} = \frac{t}{k}$$

so we have $\frac{t}{k} \le s$ that is $t \le ks$

Since $ks \le t$ and $t \le ks$ we conclude that $t = ks$

$\therefore$ by theorem 19.14

$$\limsup k s_n = k \limsup s_n$$

and since $k \limsup s_n = \infty \Rightarrow \limsup k s_n = \infty$ $\square$

Ex 19.19

Prove that every sequence has a monotone subsequence

- Let's call the "n" term of a sequence DOMINANT if it is greater than every term following it.

Note that s_n may have finitely many or infinitely many dominant terms

1) Suppose s_n has ∞-tly many dominant terms

Form a subsequence s_{n_k} solely of dominant terms of s_n. Then $s_{n_{k+1}} < s_{n_k}$ by definition of dominant

$\therefore$ s_{n_k} is a decreasing monotone subsequence of s_n

2) Suppose s_n has finitely many dominant terms

Select n_1 s.t. n_1 is beyond the last dominant term. Since n_1 is not dominant there must be some $m > n_1$ s.t. $s_m > s_{n_1}$

Select this m and call it n_2. However n_2 is still not dominant so there must be $n_3 > n_2$ with $s_{n_3} > s_{n_2}$ and so on.

The resulting sequence s_1, s_2, s_3 .. is monotone □

S — 5 SQUARES
S — 5 SQUARES
S — 5 SQUARES
S — FILL

Ex 20.13

f, g, h be functions from D into $\mathbb{R}$ and let c be an accumulation point of D.

If $f(x) \le g(x) \le h(x)$ $\forall x \in D$ with $x \ne c$ and if $\lim_{x \to c} f(x) = \lim_{x \to c} h(x) = L$ then $\lim_{x \to c} g(x) = L$

Assume $f(x) \le g(x) \le h(x)$ and $\lim_{x \to c} f(x) = \lim_{x \to c} h(x) = L$. Then

$\forall \varepsilon > 0$, $\exists \delta_1 > 0$ such that $L-\varepsilon < f(x) < L+\varepsilon$ or

$|f(x) - L| < \varepsilon$ if $x \in D$ and $0 < |x-c| < \delta_1$

$\forall \varepsilon > 0$, $\exists \delta_2 > 0$ such that $L - \varepsilon < h(x) < L + \varepsilon$ or

$|h(x) - L| < \varepsilon$ if $x \in D$ and $0 < |x-c| < \delta_2$

Let $\delta = \min(\delta_1, \delta_2)$.

If $x \in D$ and $0 < |x-c| < \delta$, then

$L - \varepsilon < f(x) \le g(x) \le h(x) < L + \varepsilon$

Thus $|g(x) - L| < \varepsilon$ when $x \in D$ and $0 < |x-c| < \delta$.

20.14 $f: D \to \mathbb{R}$ and let c be an accumulation point of D.

$a \le f(x) \le b$ $\forall x \in D$ with $x \ne c$ and $\lim_{x \to c} f(x) = L$

Prove $a \le L \le b$

→ By theorem 20.8

$a \le f(x) \le b \implies a \le f(s_n) \le b$

↓

By ex 17.18 if s_n is a convergent sequence with $a \le s_n \le b$ then $a \le \lim s_n \le b$

So we have $a \le \lim f(s_n) \le b$

and by theorem 20.8 $\lim f(s_n) = L$

$\therefore$ $a \le L \le b$

Ex 20.20

f defined on a deleted neighborhood of c

Prove: $\lim_{x \to c} f(x) = L$ iff $\lim_{x \to c^+} f(x) = \lim_{x \to c^-} f(x) = L$

Pf

Let's show first that the one sided limits exist whenever the 2 sided limit does

$(\to)$

Let $\varepsilon > 0$. If $\lim_{x \to c} f(x) = L$ we can choose $\delta > 0$ so that $|f(x) - L| < \varepsilon$ whenever $0 < |x - c| < \delta$

But then $|f(x) - L| < \varepsilon$ whenever $c - \delta < x < c$ and whenever $c < x < c + \delta$.

Thus $\lim_{x \to c^-} f(x) = L$ & $\lim_{x \to c^+} f(x) = L$

$(\leftarrow)$ Conversely

Suppose that $\lim_{x \to c^-} f(x) = L$ & $\lim_{x \to c^+} f(x) = L$ and let $\varepsilon > 0$.

We can find $\delta_1 > 0$ s.t. $|f(x) - L| < \varepsilon$ whenever $c - \delta_1 < x < c$ and find $\delta_2 > 0$ s.t. $|f(x) - L| < \varepsilon$ whenever $c < x < c + \delta_2$

$\downarrow$

Take $\delta = \min\{\delta_1, \delta_2\}$. If $0 < |x-c| < \delta$, then

either $c - \delta_1 \leq c - \delta < x < c$ or $c < x < c+\delta \leq c+\delta_2$

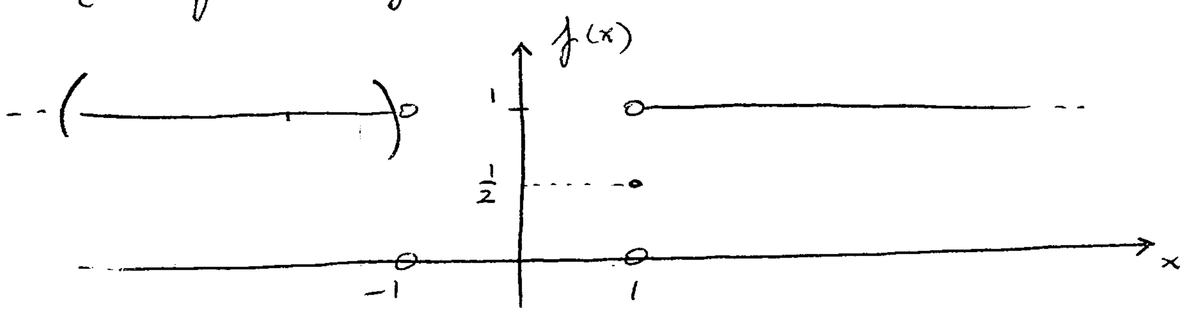
and so $|f(x) - f(c)| < \varepsilon$

Hence $\lim\limits_{x \to c} f(x) = L$ $\square$

Ex 21.8

$$f(x) = \lim_{n\to\infty} \frac{x^n}{1+x^n}$$

$D = \{x : f(x) \in \mathbb{R}\}$



$x \in (-\infty, -1) \Rightarrow f(x) = 1$

$x \in (-1, 1) \Rightarrow f(x) = 0$

$x \in (1, \infty) \Rightarrow f(x) = 1$

$f(x)$ is continuous in the intervals:

$(-\infty, 1)$ $(-1, 1)$ & $(1, \infty)$

At $x=1$ there is a jump s.t.

$\lim\limits_{n\to 1^+} \frac{x^n}{1+x^n} = 1$; $\lim\limits_{n\to 1^-} \frac{x^n}{1+x^n} = 0$ with $x=1$

but $\lim\limits_{n\to 1} \frac{x^n}{1+x^n} = \frac{1}{2}$ for $x=1$

$\therefore$ at 1 there is a discontinuity.

Ex 21.13

$f: D \to \mathbb{R}$ continuous at $c \in D$, $f(c) > 0$

Prove $\forall\ \alpha > 0$ and a neighborhood U of c s.t.

$f(x) > \alpha \quad \forall\ x \in U \cap D$

Let's assume $\alpha = \frac{1}{2} f(c)$. We know $f(c) > 0$

so $\alpha > 0$

Now, $\forall\ \varepsilon > 0$, $\exists\ \delta(\varepsilon) > 0$ s.t. $|f(x) - f(c)| < \varepsilon$

whenever $|x - c| < \delta$ and $x \in D$

Since $\alpha > 0$, let's choose $\varepsilon = \alpha = \frac{1}{2} f(c)$

$$-\frac{1}{2} f(c) < f(x) - f(c) < \frac{1}{2} f(c)$$

$$\hookrightarrow \underbrace{\frac{1}{2} f(c)}_{\alpha} < f(x) < \frac{3}{2} f(c)$$

so $\quad f(x) > \alpha$

and this is true for all $x \in U \cap D$ where U is defined as $|x - c| < \delta \quad x \in D$

Ex 21.16

$f : \mathbb{R} \to \mathbb{R}$

f is continuous on $\mathbb{R}$ iff $f^{-1}(H)$ is a closed set whenever H is a closed set.

Pf

$\mathbb{R} = (\mathbb{R} \setminus H) \cup H$

$\mathbb{R} = f^{-1}(\mathbb{R}) = f^{-1}(\mathbb{R} \setminus H) \cup f^{-1}(H)$

1) H is closed

$\Rightarrow \mathbb{R} \setminus H$ is open

So $f^{-1}(\mathbb{R} \setminus H)$ is open by corollary 21.15

$\Rightarrow \mathbb{R} \setminus f^{-1}(H)$ is open

$\therefore f^{-1}(H)$ is closed

2) Now let's assume $f^{-1}(H)$ is closed

$\Rightarrow \mathbb{R} \setminus f^{-1}(H)$ is open

So by corollary 21.15 $f^{-1}(\mathbb{R} \setminus H)$ is open

$\Rightarrow \mathbb{R} \setminus H$ is open

$\therefore H$ is closed

Ex 22.3 $f: D \to \mathbb{R}$ continuous

a) if D is open $\to$ $f(D)$ is open

FALSE

Let's take $D = \left(-\frac{3\pi}{2}, \frac{3\pi}{2}\right)$ and $f(D) = \sin x$

$f(D) = \sin x = [-1, 1]$ and it is closed while D is open ✓

e) if D is not compact, then $f(D)$ is not compact

FALSE

A set is compact iff it is closed and bounded

We can use again the same example

$D = \left(-\frac{3\pi}{2}, \frac{3\pi}{2}\right)$ is not compact

$f(D) = \sin x = [-1, 1]$ and it is both closed and bounded so compact ✓

g) if D is finite then $f(D)$ is finite

TRUE

A function is defined as a mapping for all elements in D to some elements in the range (7.1 definition)

Since D is finite $f(D)$ cannot have more elements than D, $\to$ $f(D)$ has at most the same amount of elements as in D ✓

h) if D is infinite then $f(D)$ is infinite : FALSE

In example: $D = \mathbb{R}$, $f(x) = 2$ ✓

Ex 22.4 3.5/4

$2^x = 3x$ for some $x \in (0,1)$

Let $f(x) = 2^x - 3x$. f continuous on $[0,1]$

$f(0) = 1$

$f(1) = -1$

By the intermediate value theorem, there is some $c \in (0,1)$ s.t. $f(c) = 0$

$\therefore \quad 2^c = 3c$

Ex 22.7

$f: [a,b] \to [a,b]$ is continuous

Prove f has a fixed point $\equiv \exists\, c \in [a,b]$ s.t. $f(c) = c$

Define a continuous function $g: [a,b] \longrightarrow \mathbb{R}$ by $g(x) = f(x) - x$, since $f(x)$ is continuous and the identity function is continuous. Observe $a \le f(a) \le b$ and $a \le f(b) \le b$, so $0 \le f(a) - a = g(a) \le b - a$ and $a - b \le f(b) - b = g(b) \le 0$. If $g(b) = 0$ or $g(a) = 0$ we are done. Suppose $g(a) \neq 0$ and $g(b) \neq 0$. Then by IVT $\exists\, c \in (a,b)$ with $g(c) = 0$, i.e. $f(c) = c$.

22.9 If $f: [a,b] \longrightarrow \mathbb{R}$ is continuous and $f([a,b]) \subseteq \mathbb{Q}$. Show f is constant.

Suppose f is not constant. Then For some $x_1, x_2 \in [a,b]$ $f(x_1) \neq f(x_2)$. For any pair of reals, $\exists$ an irrational, r, between them. By IVT $\exists c$ between x_1 and x_2 with $f(c) = r$ if f is continuous. This contradicts $f([a,b]) \subseteq \mathbb{Q}$.