23.3a) $f(x) = \frac{e^x}{x}$ on $[2,5]$ <u>is</u> uni. cont.

e^x is cont for $x \in \mathbb{R}$ $\quad \frac{1}{x}$ is cont. on $x \in \mathbb{R}/\{0\}$

so $\frac{e^x}{x}$ is cont. on $x \in \mathbb{R}/\{0\}$ $\quad [2,5] \subseteq \mathbb{R}/\{0\}$ and

$[2,5]$ is a compact domain so $\frac{e^x}{x}$ is unic cont.

e) $f(x) = \frac{1}{x^2}$ on $(0,1)$ <u>not</u> uni. cont. $\quad f(x)$ is uni. cont. on (a,b) iff f is cont. on $[a,b]$ but $\frac{1}{x^2}$ is not cont at $x=0$

23.5) $f(x) = \sqrt{x}$ is uni. cont on $[0,\infty)$

f is cont. on $[0,2]$ ∈ compact so f is uni. cont. on $[0,2]$

for $(1,\infty)$ $\quad |\sqrt{x} - \sqrt{y}| < \varepsilon$ when $|x-y| < \delta$ and $\sqrt{x}, \sqrt{y} > 1$ so

$$|\sqrt{x} - \sqrt{y}| = \frac{|\sqrt{x} + \sqrt{y}|\,|\sqrt{x} - \sqrt{y}|}{|\sqrt{x} + \sqrt{y}|} = \frac{|x-y|}{|\sqrt{x} + \sqrt{y}|} \le |x-y| \le \delta$$

let $\varepsilon = \delta$ then if $|x-y| \le \delta$ $\quad |\sqrt{x} - \sqrt{y}| \le \delta = \varepsilon$

so $f(x)$ is uni. cont on $(1,\infty)$ then

let δ_1 be the delta for $[0,2]$ and δ_2 for $(1,\infty)$

$\delta_3 = \min(\delta_1, \delta_2)$

if $|x-y| < \delta_3$, we either have

1) $x, y \in [0,2]$
and $|x-y| < \delta_3 \le \delta_1 \Rightarrow |\sqrt{x} - \sqrt{y}| < \varepsilon$.

2) $x, y \in (1,\infty)$
and $|x-y| < \delta_3 \le \delta_2 \Rightarrow |\sqrt{x} - \sqrt{y}| < \varepsilon$.

3) $x \in [0,2]$ & $y \in (1,\infty)$

but since $\delta_3 < 1$

$\Rightarrow$ $x \in (1,2] \Rightarrow$ $x, y \in (1,\infty)$

& by case 2 $|\sqrt{x} - \sqrt{y}| < \varepsilon$

or $y \in (x, x+1) \subseteq [0,2]$

& by case 1 $|\sqrt{x} - \sqrt{y}| < \varepsilon$

and $f(x)$ uniformly continuous.

23.15 Since $f(x) = f(x+k)$ $\forall x \in \mathbb{R}$, restrict $f(x)$ to $[0,k]$. $f(x)$ is continuous on $[0,k]$, which is compact and hence $f(x)$ is bounded on $[0,k]$. Now since $f(x)$ is bounded on $[0,k]$, if $x \in \mathbb{R}$, $\exists n \in \mathbb{Z}$ with $x+nk \in [0,k]$. Thus $|f(x+nk)| = |f(x)| < M$, the bound for $f(x)$ on $[0,k]$.

Again since $f(x)$ is continuous on $[0,k]$, $f(x)$ is uniformly continuous on $[0,k]$. In fact, $f(x)$ is uniformly continuous on $[(n-1)k, nk]$ $\forall n \in \mathbb{Z}$ with the same δ, ie $\forall \varepsilon, \exists \delta$ if $|x-y| < \delta$ & $x, y \in [0,k]$ then $|f(x) - f(y)| < \varepsilon/2$

note if $x, y \in [(n-1)k, nk]$ & $|x-y| < \delta \Rightarrow |x-(n-1)k - (y-(n-1)k)| < \delta$

but $|f(x-(n-1)k) - f(y-(n-1)k)| = |f(x) - f(y)| < \varepsilon/2$

Note, if $|x-y| < \delta$, then either $(n-1)k < x, y < nk$ some $n \in \mathbb{Z}$

or $x < nk < y$ some n (or $y < nk < x$ some n)

$|x-y| \le |x-nk| + |nk-y| < 2\delta$ $\Rightarrow$ $|f(x) - f(nk)| < \varepsilon/2$

& $|f(nk) - f(y)| < \varepsilon/2 \Rightarrow |f(x) - f(y)| < \varepsilon$

and $f(x)$ is uniformly continuous for all reals

23. 6) $\forall \varepsilon > 0 \; \exists \delta_1 > 0$: $|x-y| < \delta_1$ implies $|f(x) - f(y)| < \varepsilon/2$

$\forall \varepsilon > 0 \; \exists \delta_2 > 0$; $|x-y| < \delta_2$ implies $|g(x) - g(y)| < \varepsilon/2$

need to show $\forall \varepsilon, \exists \delta_3$ $|x-y| < \delta_3$ implies $|f(x) + g(x) - f(y) - g(y)| < \varepsilon$

$2|x-y| < \delta_1 + \delta_2$ implies $|f(x) - f(y)| + |g(x) - g(y)| < \varepsilon/2 + \varepsilon/2 = \varepsilon$

but $|f(x) + g(x) - f(y) - g(y)| \le |f(x) - f(y)| + |g(x) - g(y)| < \varepsilon/2 + \varepsilon/2 = \varepsilon$

let $\delta_3 = \frac{\delta_1 + \delta_2}{2}$ then $|x-y| < \delta_3$ implies

$|f(x) + g(x) - f(y) - g(y)| \le \varepsilon/2 + \varepsilon/2 = \varepsilon$ so $f+g$ is uni. cont.

23.10) $f(x) = g(x) = x$ $f(x), g(x)$ are uniformally cont. on $\mathbb{R}$

but $fg(x) = x^2$ is not

see attached for 23.15

25.6a $f(x) = x^2 \sin(1/x)$ $x \ne 0$ and $f(0) = 0$ which is well defined

$f'(x) = 2x \sin(1/x) - \cos(1/x)$

except at $x = 0$

b) $\lim_{x \to c} \frac{f(x) - f(c)}{x - c} \Rightarrow \lim_{x \to 0} \frac{f(x) - f(0)}{x} = \lim_{x \to 0} \frac{f(x)}{x} = \lim_{x \to 0} x \sin(1/x)$

but $-1 \le \sin(1/x) \le 1$ so $\lim_{x \to 0} x \sin(1/x) = 0$ so $f'(0) = 0$

since $\lim_{x \to 0} |x \sin(1/x)| \le \lim_{x \to 0} |x| = 0$

25.6 c) $\lim_{x\to 0} 2x\sin(1/x) - \cos(1/x)$

notice that the first part goes to 0 since

$\lim_{x\to 0} |2x\sin(1/x)| \leq \lim |2x| = 0$

but the second part osscilates between -1 and 1 so the limit can not exist. So f' is not cont at $x=0$

25.8) $f(x) = x^2\sin(1/x^2)$ for $x \neq 0$ $f(0)=0$

a) $f'(x) = 2x\sin(1/x^2) - \dfrac{2\cos(1/x^2)}{x}$ which is well defined

except for possibly at $x=0$ so f is diff on $x \in \mathbb{R}/\{0\}$

for $x=0$ $\lim_{x\to 0} \dfrac{f(x)-f(0)}{x-0} = \lim x\sin(1/x^2) \leq \lim_{x\to 0} x = 0$

$\geq -\lim_{x\to 0} x = 0$

so f is also diff at $x=0$ and $f'(0)=0$

b) $\lim_{x\to 0} f'(x) = \lim 2x\sin(1/x^2) - \dfrac{2\cos(1/x^2)}{x} = \lim \dfrac{\cos(1/x^2)}{x}$

the $\cos(1/x)$ will keep oscilating between 1 and -1 while the denominator gets smaller blowing it up so limit does not exist.

25.11) $f(x) = x^2$ if $x \in \mathbb{Q}$ and $f(x)=0$ if x is irrational

a) let $x \neq 0$ be a rational then there exists a sequence of irrationals approaching x since rationals are dense in reals. But the value at all the sequence points is 0 so if we make ε less then the value at x we can find a δ arbitraryly small wich would make $f(x) - f(x+\delta) > \varepsilon$ since we can make $f(x+\delta)$ be an irrational $f(x) - 0 > \varepsilon$

at $x=0$ however ~~the ε we choose~~ let $\delta = \min(\varepsilon^{1/2}, 1)$ then

$|x-0| < \delta = \varepsilon^{1/2}, 1$ has to imply $|f(x)-0| < \varepsilon$

but $|f(x)-0| \le x^2 \le \delta^2 = \varepsilon$

and for x being an irrational we can find a rational arbitrary close to it ~~and the minimum f(x) f(x+δ)~~ and make the same argument as for the rationals.

b) suppose f differentiable at $x \neq 0$

If x rational, then there exist a sequence x_n of irrational #s with $\lim_{n\to\infty} x_n = x$. f diff at x if $\lim_{n\to\infty} \frac{f(x)-f(x_n)}{x-x_n}$ exists. But $\lim_{n\to\infty} \frac{f(x)-f(x_n)}{x-x_n} = \lim_{n\to\infty} \frac{x^2}{x-x_n}$ which does not exist, as the numerator is fixed and the denominator becomes increasingly small.

If x irrational, there exists a sequence x_n of rationals with $\lim_{n\to\infty} x_n = x$. f diff at x if $\lim_{n\to\infty} \frac{f(x)-f(x_n)}{x-x_n}$ exists. But $\lim_{n\to\infty} \frac{f(x)-f(x_n)}{x-x_n} = \lim_{n\to\infty} \frac{-x_n^2}{x-x_n} = \lim_{n\to\infty} \frac{x^2}{x-x_n}$ which also doesn't exist.

However if $x=0$

formally $\lim_{x\to c} \frac{f(x)-f(c)}{x-c} = \lim_{x\to 0} \frac{f(x)-f(0)}{x} =$

$\lim_{x\to 0} \frac{f(x)}{x} = \lim \frac{x^2}{x} = 0$

or $\lim \frac{0}{x} = 0$

so f is diff at $x=0$

25.12) $(x-a)^2 | P(x)$ then $\frac{P(x)}{(x-a)^2} = Q(x)$ where $Q(x)$ is a polynomial

$P(x) = (x-a)^2 Q(x) \Rightarrow P'(x) = 2(x-a)Q(x) + (x-a)^2 Q'(x)$

polynomials are closed under differentiation, ~~and~~ multiplication and addition so $2Q(x) + (x-a)Q'(x)$ is also a polynomial

$(x-a)[2Q(x) + (x-a)Q'(x)] = \cancel{(x-a)P(x)}\ (x-a)G(x) = P'(x)$

and $(x-a) | (x-a)(G(x))$ is true so $(x-a) | P'(x)$ is true

26.6) $f(x) = x$ is cont. and diff. but $f'(x) = 1 \neq 0$

$f(x) = |x|$ on $[-a, a]$ is cont and $f(-a) = f(a)$ but

$f'(x) = \begin{cases} 1 & \text{for } x > 1 \\ -1 & \text{for } x < 1 \end{cases}$

$f(x) = x^3$ on $[1, 2]$ f is cont and diff but

$f'(x) = 3x^2$ $\quad 3 \leq f'(x) \leq 12$ so $f'(x) \neq 0$ for $x \in [1, 2]$

26.9) a) if $f'(x) > 0$ $\forall x \in I$ then f is strictly increasing

b) $f'(x) < 0$ .... f is strictly decreasing

converse if f is strictly increasing then $f'(x) > 0$ $\forall x \in I$

a) let $f(x) = x^3$ then $\forall x, y \in \mathbb{R} : x < y$ implies $x^3 < y^3$

but $f'(x) \not> 0$ since $f'(0) = 0$

b) let $f(x) = -x^3$ .... implies $x^3 > y^3$

but $f'(x) \not< 0$ since $f'(0) = 0$

26.11) f is diff on $[a,b]$. $f'(x) \geq 0 \quad \forall x \in [a,b]$

and $f'(A) = 0 \quad \forall A \subseteq [a,b]$ show f is strictly inc.

From problem 26.8 we know that f is increasing.

Also since $f'(A) = 0$ can only be true if A has measure 0

so $f(x) < f(y)$ if $x < y$. Since if $f(x) = f(y)$ and

$x < y$ then by IVT either $f'(z) = 0 \quad \forall x < z < y$

which was not allowed or $f'(z) < 0$ which is also not

allowed by definition of problem.

f is diff on $\mathbb{R}$

26.13) a) $f(0) = 0$, $f(1) = 2$ $f(2) = 2$

show $\exists c_1 \in (0,1) : f'(c_1) = 2$ automatic from IVT

b) show $\exists c_2 \in (1,2) : f'(c_2) = 0$ automatic from Rolle's Theorm

c) show $\exists c_3 \in (0,2) : f'(c_3) = \frac{5}{4}$ automatic from IVT for Derivatives

and $0 \leq \frac{5}{4} \leq 2$

26.23a) $|f(x) - f(y)| \leq M|x-y|^{\alpha}$ if $|x-y| < \delta$

~~if~~ then $|x-y|^{\alpha} < \delta^{a}$ let $\delta^{a} = \frac{\varepsilon}{M}$

then $|f(x) - f(y)| \leq M\delta^{a} = \varepsilon$ so $|x-y| < \delta$ ~~implies~~

~~for ... that ... holds~~ implies $|f(x) - f(y)| < \varepsilon$ so

f is uni cont.

b)

c) $a = 1$ then f is not neccessarily diff.

$f = |x|$ is Lipschitz since $||x| - |y|| \leq M|x-y|$

but not diff at $t = 0$

d)

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26.23) b) $\lim_{x \to c} \frac{|f(x) - f(c)|}{x - c} \leq \lim M \frac{|x-c|^{\alpha}}{x-c} = \lim M |x-c|^{\alpha - 1}$ for $\alpha > 1$

$= 0$ so $f'(x) = 0$ $\forall x$ if $\alpha > 1$ so constant.

d) if $\lim_{x \to c} \frac{g(x) - g(c)}{x - c} \leq M$ $\forall x$ so

$|g(x) - g(c)| \leq |x - c| M$ since the lim was true $\forall x$