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The cl-core of an ideal

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Abstract

We expand the notion of core to cl-core for Nakayama closures cl. In the characteristic $p > 0$ setting, when cl is the tight closure, denoted by $*$, we give some examples of ideals when the core and the $*$-core differ. We note that $*$-core$(I) =$ core(I), if I is an ideal in a one-dimensional domain with infinite residue field or if I is an ideal generated by a system of parameters in any Noetherian ring. More generally, we show the same result in a Cohen–Macaulay normal local domain with infinite perfect residue field, if the analytic spread, ℓ, is equal to the $*$-spread and I is G_ℓ and weakly-$(\ell - 1)$-residually S_2. This last is dependent on our result that generalizes the notion of general minimal reductions to general minimal $*$-reductions. We also determine that the $*$-core of a tightly closed ideal in certain one-dimensional semigroup rings is tightly closed and therefore integrally closed.

1. *Introduction*

The core of an ideal, the intersection of all reductions of the ideal, was introduced by Rees and Sally in [**24**] in the 80's. Then over a decade passed before Huneke and Swanson [**13**] analyzed the core of ideals in 2-dimensional regular local rings. Then a stream of papers came out within a decade by Corso, Polini and Ulrich [**4**], [**5**], [**23**], Hyry and Smith [**17**], [**18**] and Huneke and Trung [**15**] expanding the understanding and computability of the core. As it is the intersection of reductions, in general it lies deep within the ideal. In fact, the core is related to the *Briançon–Skoda* Theorem [**20**]: let R be a regular local ring of dimension d and let I an ideal. Then $\overline{I^d} \subseteq J$ for any reduction J of I. Hence $\overline{I^d} \subseteq \text{core}(I)$. A very slick proof of the Briançon–Skoda Theorem was given in characteristic $p > 0$, using tight closure, [**10**, theorem 5·4]. We expand the notion of core to other closure operations; in particular, Nakayama closure operations. Epstein defined the Nakayama closure as follows:

Definition 1·1. ([**7**]) A closure operation cl, defined on a Noetherian local ring $(R, \mathfrak{m})$ is a Nakayama closure if for all ideals I and J satisfying $J \subset I \subset (J + \mathfrak{m}I)^{cl}$ it follows that $I \subset J^{cl}$.

Note that integral closure, tight closure and Frobenius closure are examples of Nakayama closures, [**7**, proposition 2·1]. Recall that both the tight closure and the Frobenius closure are characteristic $p > 0$ notions.

Epstein's main reason for the definition of Nakayama closure was to expand the notion of reduction and analytic spread to these other closure operations. With a well defined notion of reduction and analytic spread, we can easily extend the definition of the core to these other closure operations. In general, the cl-cores will not lie as deep in the ideal as the core itself. This will follow from the fact that the partial ordering of closure operations leads to a reverse partial ordering on the cl-cores (Proposition 3·4). Our hope in studying these cl-cores is that tight closure methods may be used to compute the core in situations where the core and the $*$-core agree.

In Section 2, we provide some background information about the core and tight closure theory, along with a review of some central theorems that are used in this article. In Section 3, we review cl-reductions of ideals. We also discuss the cl-spread of an ideal and define both the cl-deviation and the second cl-deviation in terms of the cl-spread. We also introduce the notion of cl-core. In Section 4, we show different instances when the core and the $*$-core agree. Our main result, Theorem 4·5, shows that we can form general minimal $*$-reductions. This allows us to show in particular that if $(R, \mathfrak{m})$ is a Gorenstein normal local isolated singularity of positive characteristic with infinite perfect residue field, test ideal equal to $\mathfrak{m}$ and I is an $\mathfrak{m}$-primary tightly closed ideal then $*\text{-core}(I) = \text{core}(I)$ (Corollary 4·7). Also, when $(R, \mathfrak{m})$ is a Cohen–Macaulay normal domain of positive characteristic and infinite perfect residue field and I is an ideal that satisfies G_ℓ and is weakly $(\ell - 1)$-residually S_2 with $\ell^*(I) = \ell(I) = \ell$ then $\text{core}(I) = *\text{-core}(I)$ (Theorem 4·8). In Section 5, we discuss when the $*$-core is tightly closed in some one-dimensional semigroup rings. Finally, in Section 6, we give some examples where we compute the $*$-core and in each case we compare the core with the $*$-core.

2. *Background*

In this section we recall some definitions and results that we will use extensively in this paper.

Definition 2·1. Let R be a Noetherian local ring of characteristic $p > 0$. We denote positive powers of p by q and the set of elements of R which are not contained in the union of minimal primes by R^0. Then

(a) For any ideal $I \subset R$, $I^{[q]}$ is the ideal generated by the qth powers of elements in I.

(b) We say an element $x \in R$ is in the *tight closure*, I^*, of I if there exists a $c \in R^0$, such that $cx^q \in I^{[q]}$ for all large q.

(c) We say an element $x \in R$ is in the *Frobenius closure*, I^F, of I if $x^q \in I^{[q]}$ for all large q.

Finding the tight closure of an ideal would be hard without test elements and test ideals. A *test element* is an element $c \in R^0$ such that $cI^* \subset I$ for all $I \subset R$. We note here that $c \in \bigcap_{I \subset R} (I : I^*)$. Since the intersection of ideals is an ideal we call the ideal $\tau = \bigcap_{I \subset R} (I : I^*)$ *the test ideal* of R, namely the ideal generated by all the test elements. We say that I is a *parameter ideal* if I is generated by part of a system of parameters. In a Gorenstein local isolated singularity, the following theorem of Smith [**25**] gives a nice way to compute the tight closure of a parameter ideal using the test ideal.

THEOREM 2·2 ([**25**, lemma 3·6, proposition 4·5]). *Let R be a Gorenstein local isolated singularity with test ideal τ. Then for any system of parameters $x_1, x_2, \ldots, x_d$,*

$$(x_1, x_2, \ldots, x_d) : \tau = (x_1, x_2, \ldots, x_d)^*.$$

Related concepts are parameter test elements and parameter test ideals. A *parameter test element* is an element $c \in R^0$ such that $cI^* \subset I$ for all parameter ideals $I \subset R$. Note that $c \in \bigcap_{I \subset R} (I : I^*)$. Let $P(R)$ be the set of parameter ideals in R. We call $\tau_{par} = \bigcap_{I \in P(R)} (I : I^*)$ *the parameter test ideal.* It is known in a Gorenstein ring that $\tau = \tau_{par}$. We can relax the Gorenstein assumption from the above theorem and obtain:

THEOREM 2·3 ([**27**]). *Let R be a Cohen–Macaulay local isolated singularity with parameter test ideal τ_{par}. For any system of parameters $x_1, x_2, \ldots, x_d$,*

$$(x_1, x_2, \ldots, x_d) : \tau_{par} = (x_1, x_2, \ldots, x_d)^*.$$

Note, if the test ideal is known to be $\mathfrak{m}$, where $\mathfrak{m}$ is the maximal ideal of the ring, then the parameter test ideal will also be $\mathfrak{m}$.

Another result that we will use repeatedly is the following due to Aberbach:

PROPOSITION 2·4 ([**1**, proposition 2·4]). *Let $(R, \mathfrak{m})$ be an excellent, analytically irreducible local ring of characteristic $p > 0$, let I be an ideal, and let $f \in R$. Assume that $f \notin I^*$. Then there exists $q_0 = p^{e_0}$ such that for all $q \geqslant q_0$ we have $I^{[q]} : f^q \subset \mathfrak{m}^{[q/q_0]}$.*

Notice that later on we will be assuming that R is an excellent normal local domain, which implies that R is analytically irreducible, since the completion of an excellent normal domain is again a normal domain. Hence one may use Proposition 2·4.

Let R be a Noetherian ring and I an ideal. We say that $J \subset I$ is a *reduction* of I if $I^{n+1} = JI^n$ for some nonnegative integer n. Northcott and Rees introduced this notion in [**22**] in order to study multiplicities. If $(R, \mathfrak{m})$ is a Noetherian local ring and I is an $\mathfrak{m}$-primary ideal then I and its reduction J have the same multiplicity and thus one may want to shift the attention from I to the simpler ideal J. If R is a Noetherian local ring with infinite residue field then I has infinitely many minimal reductions or I is basic, i.e. I is the only reduction of itself ([**22**]). When R is a Noetherian local ring and I is an ideal then a reduction J of I is called minimal if it is minimal with respect to inclusion. To facilitate the lack of uniqueness for minimal reductions, Rees and Sally introduced the core of an ideal:

Definition 2·5 ([**24**]). Let R be a Noetherian ring. Let I be an ideal. Then $\operatorname{core}(I) = \bigcap_{J \subset I} J$, where J is a reduction of I.

When R is a Noetherian local ring it is enough to take the intersection over all minimal reductions since every reduction contains a minimal reduction. There has been a significant effort by several authors to find efficient ways of computing this infinite intersection. One result in particular is of special interest to us.

THEOREM 2·6 ([**4**, theorem 4·5]). *Let R be a Cohen–Macaulay local ring with infinite residue field and I an ideal of analytic spread ℓ. Assume that I satisfies G_ℓ and is weakly $(\ell - 1)$-residually S_2. Then $\operatorname{core}(I) = \mathfrak{a}_1 \bigcap \cdots \bigcap \mathfrak{a}_t$ for $\mathfrak{a}_1, \ldots, \mathfrak{a}_t$ general ℓ-generated ideals in I which are reductions of I and for some finite integer t.*

We now explain the conditions in the statement of Theorem 2·6.

The *analytic spread* of I, $\ell(I)$, is the Krull dimension of $\mathcal{F}(I) := \bigoplus_{i\geqslant 0} I^i/\mathfrak{m}I^i$, the special fiber ring of I. It is well known that if R is a Noetherian local ring with infinite residue field then any minimal reduction J of I has the same minimal number of generators, namely $\mu(J) = \ell(I)$ [**22**]. It is straightforward to see that in general ht $I \leqslant \ell(I) \leqslant \dim R$.

Following the definitions given in [**3**] we say that an ideal I satisfies the property G_s if for every prime ideal $\mathfrak{p}$ containing I with $\dim R_\mathfrak{p} \leqslant s-1$, the minimal number of generators, $\mu(I_\mathfrak{p})$, of $I_\mathfrak{p}$ is at most $\dim R_\mathfrak{p}$. A proper ideal K is called an *s-residual intersection* of I if there exists an s-generated ideal $\mathfrak{a} \subset I$ so that $K = \mathfrak{a} : I$ and ht $K \geqslant s \geqslant$ ht I. If ht $I + K \geqslant s+1$, then K is said to be a *geometric s-residual intersection* of I. If R/K is Cohen–Macaulay for every i-residual intersection (geometric i-residual intersection) K of I and every $i \leqslant s$ then I satisfies AN_s (AN_s^-). An ideal I is called *s-residually S_2* (*weakly s-residually S_2*) if R/K satisfies Serre's condition S_2 for every i-residual intersection (geometric i-residual intersection) K of I and every $i \leqslant s$.

Remark 2·7. Let $(R, \mathfrak{m})$ be a Noetherian local ring and I an ideal. Let $g =$ ht I. It is not difficult for an ideal to satisfy the condition G_s. If I is an $\mathfrak{m}$-primary ideal or in general an equimultiple ideal, i.e. $\ell = \ell(I) =$ ht I, then I satisfies G_ℓ automatically.

If $(R, \mathfrak{m})$ is a Cohen–Macaulay local ring of dimension d and I an ideal satisfying G_s, then I is universally s-residually S_2 in the following cases:

(a) R is Gorenstein, and the local cohomology modules $H_\mathfrak{m}^{d-g-j}(R/I^j)$ vanish for all $1 \leqslant j \leqslant s-g+1$, or equivalently, $Ext_R^{g+j}(R/I^j, R) = 0$ for all $1 \leqslant j \leqslant s-g+1$ ([**3**, theorem 4·1 and 4·3]).

(b) R is Gorenstein, depth $R/I^j \geqslant \dim\ R/I - j + 1$ for all $1 \leqslant j \leqslant s-g+1$ ([**26**, theorem 2·9(a)]).

(c) I has sliding depth ([**9**, theorem 3·3]).

Notice that condition (b) implies (a) and the property AN_s by [**26**, theorem 2·9(a)]. Also the conditions (b) and (c) are satisfied by *strongly Cohen–Macaulay* ideals, i.e. ideals whose Koszul homology modules are Cohen–Macaulay. If I is a Cohen–Macaulay almost complete intersection or a Cohen–Macaulay deviation two ideal of a Gorenstein ring then I is strongly Cohen–Macaulay ([**2**, p. 259]). Furthermore, if I is in the linkage class of a complete intersection (licci) then I is again a strongly Cohen–Macaulay ideal ([**11**, theorem 1·11]). Standard examples include perfect ideals of height two and perfect Gorenstein ideals of height three.

3. *cl-Reductions and the definition of cl-core*

Let R be a Noetherian ring and I an ideal. Recall that $J \subset I$ is a reduction of an ideal I if $JI^n = I^{n+1}$ for some nonnegative integer n. If J is a reduction of I, then $J \subset I \subset \overline{J}$. Epstein defines a cl-reduction of an ideal I to be an ideal J such that $J \subset I \subset J^{cl}$. If cl is a Nakayama closure we have the following Lemma:

LEMMA 3·1 ([**7**, lemma 2·2]). *Let R be a Noetherian local ring and I an ideal. If cl is a Nakayama closure on R, then for any cl-reduction J of I, there is a minimal cl-reduction K of I contained in J. Moroever, in this situation any minimal generating set of K extends to a minimal generating set of J.*

In particular Lemma 3·1 shows that minimal cl-reductions exist. Following the idea in Definition 2·5 we now define the cl-core.

Definition 3·2. Let R be a Noetherian ring and cl a closure defined on R. The cl-core of an ideal I is $cl\text{-core}(I) = \bigcap_{J \subset I} J$, where J is a cl-reduction of I.

Recall, an ideal is basic if it does not have any nontrivial reductions. We will say that an ideal is cl-basic if it does not have any nontrivial cl-reductions. Clearly if I is a basic ideal $\text{core}(I) = I$. If I is a cl-basic ideal then $cl\text{-core}(I) = I$. Note that we can restrict the intersection to the minimal cl-reductions of I, when R is a Noetherian local ring. In [**28**] the second author has discussed the partial ordering on the set of closure operations of a ring defined as follows: If cl_1 and cl_2 are closure operations we say that $cl_1 \leqslant cl_2$ if and only if $I^{cl_1} \subset I^{cl_2}$ for all ideals I of R.

LEMMA 3·3. *Let cl_1 be a closure operation and cl_2 be a Nakayama closure operation defined on a Noetherian local ring R with $cl_1 \leqslant cl_2$. Let I be an ideal. If J_1 is a minimal cl_1-reduction of I then there exists a minimal cl_2-reduction J_2 of I with $J_2 \subset J_1$.*

Proof. Notice that $J_1 \subset I \subset J_1^{cl_1}$, as J_1 is a cl_1-reduction of I. Since $cl_1 \leqslant cl_2$ then $K^{cl_1} \subset K^{cl_2}$ for all ideals $K \subset R$. Hence $J_1^{cl_1} \subset J_1^{cl_2}$ and $J_1 \subset I \subset J_1^{cl_1} \subset J_1^{cl_2}$. So J_1 is a cl_2-reduction of I also. Now by Lemma 3·1, there is a minimal cl_2-reduction of I contained in J_1.

One consequence of Lemma 3·3 is the following:

PROPOSITION 3·4. *Let cl_1 be a closure operation and cl_2 be a Nakayama closure operation defined on a Noetherian local ring R with $cl_1 \leqslant cl_2$. Let I be an ideal. Then* $cl_2\text{-core}(I) \subset cl_1\text{-core}(I)$.

Proof. We know that $cl_1\text{-core}(I) = \bigcap_{J_1 \subset I} J_1$ where J_1 are cl_1-reductions of I. By Lemma 3·3, for every cl_1-reduction J_1 of I, there exists a minimal cl_2-reduction, J_2 contained in J_1. Thus $cl_2\text{-core}(I) \subset \bigcap_{J_2 \subset J_1} J_2$ and $\bigcap_{J_2 \subset J_1} J_2 \subset \bigcap_{J_1 \subset I} J_1 = cl_1\text{-core}(I)$.

Let R be a Noetherian ring of characteristic $p > 0$. Note that $I^F \subseteq I^* \subseteq \overline{I}$ for all ideals I of R. The first inclusion is clear as $x \in I^F$ if $x^q \in I^{[q]}$ for all $q \gg 0$ implies that $cx^q \in I^{[q]}$ for some $c \in R^o$ namely, by taking $c = 1$. The second inclusion holds by [**10**, theorem 5·2]. In particular, we have the following corollary regarding the Frobenius or F-core, the $*$-core and the core, which is a cl-core where cl is the integral closure.

COROLLARY 3·5. *Let R be an excellent local ring of characteristic $p > 0$ and let I be an ideal. Then* $\text{core}(I) \subset *\text{-core}(I) \subset F\text{-core}(I)$.

Following [**14**, proposition 17·8·9] we see:

COROLLARY 3·6. *Let R be a Noetherian local ring and let I be an ideal. Then $\sqrt{I} = \sqrt{cl\text{-core}(I)}$ for any $cl \leqslant^-$. In particular, if R is an excellent local ring of characteristic $p > 0$ it follows that $\sqrt{I} = \sqrt{*\text{-core}(I)} = \sqrt{F\text{-core}(I)}$.*

To better understand these minimal cl-reductions, Epstein mimicked Vraciu's definition of $*$-independence in [**30**] to define cl-independence. The elements $x_1, \ldots, x_n$ are said to be cl-independent if $x_i \notin (x_1, \ldots, \widehat{x_i}, \ldots, x_n)^{cl}$, for all $1 \leqslant i \leqslant n$. Then he further refines the notion to that of strong cl-independence. An ideal is strongly cl-independent if every minimal set of generators is cl-independent. Epstein then showed in [**7**, proposition 2·3] that when cl is a Nakayama closure, J is a minimal cl-reduction of I if and only if J is a strongly cl-independent ideal.

In a Noetherian local ring $(R, \mathfrak{m})$ of characteristic $p > 0$ Vraciu [**30**] defined the special tight closure, I^{*sp}, to be the elements $x \in R$ such that $x \in (\mathfrak{m}I^{[q_0]})^*$ for some $q_0 = p^{e_0}$ and some $e_0 \in \mathbb{N}$. Huneke and Vraciu show in [**16**, proposition 4·2] that $I^{*sp} \bigcap I = \mathfrak{m}I$ if I is generated by $*$-independent elements. Note that the minimal $*$-reductions of I are generated by $*$-independent elements. Epstein showed in [**7**, lemma 3·4] that $I^{*sp} = J^{*sp}$ for all $*$-reductions of I.

An ideal I is said to have cl-spread, $\ell^{cl}(I)$, if all minimal cl-reductions of I have the same minimal number of generators. As with the analytic spread, Epstein proves that $\mu(J) = \ell^{cl}(I)$ for all minimal cl-reductions J of I. He also goes on to prove [**7**, theorem 5·1] that the $*$-spread is well defined over an excellent analytically irreducible local domain of characteristic $p > 0$. Now if the cl_1- and the cl_2-spread are defined for I, we have:

PROPOSITION 3·7. *Let cl_1 be a closure operation and cl_2 be a Nakayama closure operation defined on a Noetherian local ring R with $cl_1 \leqslant cl_2$. Let I be an ideal with well-defined cl_1- and cl_2-spread. Then $\ell^{cl_1}(I) \geqslant \ell^{cl_2}(I)$.*

Proof. Let J_1 be a cl_1-minimal reduction of I. Then $\mu(J_1) = \ell^{cl_1}(I)$ [**7**, proposition 2·4]. Also $J_1 \subset I \subset J_1^{cl_1} \subset J_1^{cl_2}$, since $cl_1 \leqslant cl_2$. Therefore J_1 is also a cl_2-reduction of I (not necessarily minimal). Hence $\mu(J_1) \geqslant \ell^{cl_2}(I)$ and equality holds if and only if J_1 is a minimal cl_2-reduction of I, according to [**7**, proposition 2·4]. Hence $\ell^{cl_1}(I) = \mu(J_1) \geqslant \ell^{cl_2}(I)$.

In particular, we have the following corollary regarding the Frobenius or F-spread, the $*$-spread and the analytic spread of an ideal:

COROLLARY 3·8. *Let R be an excellent local ring of characteristic $p > 0$ and let I be an ideal. Then $\ell(I) \leqslant \ell^*(I) \leqslant \ell^F(I)$.*

The analytic spread is bounded above by the dimension of the ring, but in principle, the cl-spreads can grow arbitrarily large. The cl-spread of an ideal I is however bounded by the minimal number of generators of I, $\mu(I)$.

There are two invariants of a ring related to the analytic spread: the analytic deviation and the second analytic deviation. Recall that in a Noetherian local ring, the *analytic deviation* of an ideal I is $ad(I) = \ell(I) - \operatorname{ht} I$. Note that I is equimultiple if $ad(I) = 0$. The *second analytic deviation* of I is $ad_2(I) = \mu(I) - \ell(I)$. We make the following definitions with respect to the cl-spread of an ideal I.

Definition 3·9. Let R be a Noetherian local ring and cl a closure operation on R. Let I be an ideal with a well defined cl-spread. The cl-deviation of I is $cld(I) = \ell^{cl}(I) - \operatorname{ht} I$. The second cl-deviation of I is $cld_2(I) = \mu(I) - \ell^{cl}(I)$.

Remark 3·10. Let R be a Noetherian local ring and cl a closure operation on R. Let I be an ideal with a well defined cl-spread. Assume $cl \leqslant^{-}$. The following are straightforward from the definition above.

(a) Since $\ell(I) \leqslant \ell^{cl}(I)$ then $cld(I) \geqslant 0$.
(b) Note that in a Cohen–Macaulay local ring, if I is generated by a system of parameters then I is equimultiple and we have $cld(I) = 0$.
(c) Since $\ell(I) \leqslant \ell^{cl}(I)$, then $cld_2(I) \leqslant ad_2(I)$.

Note if I is cl-closed, then $\ell^{cl}(I)=\mu(I)$. If I is a basic ideal (i.e. $^-$-basic) and $cl \leqslant ^-$, then $\ell^{cl}(I)=\ell(I)$. We would like to know how the core(I) and the cl-core(I) are related when $\ell(I)=\ell^{cl}(I)$.

4. *When* $*$-*core and core agree*

First we record some straightforward cases when the core and the $*$-core agree.

PROPOSITION 4·1. *Let R be a Noetherian local ring of characteristic $p>0$ and I be an ideal generated by a system of parameters. Then* $*$-core$(I)=$ core(I).

Proof. An ideal generated by a system of parameters is basic and $*$-basic, hence the only reduction (and $*$-reduction) of I is I. Thus $*$-core$(I)=$ core$(I)=I$.

Note, when I is generated by a system of parameters, we may have $I^* \subsetneq \overline{I}$, but the core and the $*$-core are equal.

PROPOSITION 4·2. *Let R be a one-dimensional local domain of characteristic $p>0$ with infinite residue field and let I be an ideal. Then* $*$-core$(I)=$ core(I).

Proof. If $I=0$ then the assertion is clear. Suppose then that $I \neq 0$ then $\ell(I)=1$. By [**12**, example 1·6·2] it is known that for principal ideals $\overline{(x)}=(x)^*$ and also that $I^*=(x)^*$, for some $x \in R$. Then every minimal reduction and hence minimal $*$-reduction of I is principal. Therefore we obtain equality of the core and the $*$-core.

We would like to show that in an excellent normal local ring the core and the $*$-core agree for ideals of second $*$-deviation 1. Note that if $(R, \mathfrak{m})$ is Gorenstein local isolated singularity of characteristic $p>0$ with test ideal $\mathfrak{m}$ and I is an ideal generated by a system of parameters, then $*d_2(I)=1$ by Theorem 2·2 since the tight closure is the socle in this case, see also the proof of Corollary 4·6.

To show that the core and the $*$-core agree for ideals with $*d_2(I)=1$, we will begin by considering general minimal reductions. Recall:

Definition 4·3 ([**22**]). Let R be a Noetherian local ring with infinite residue field k. Let I be an ideal generated by $f_1, \ldots, f_m$ and let t be a fixed positive integer. We say that $b_1, \ldots, b_t$ are t general elements in I if there exists a dense open subset U of $\mathbb{A}_k^{tm}$ such that for $1 \leqslant j \leqslant m$, we have $b_i = \sum_{j=1}^{m} \lambda_{ij} f_j$, where $\underline{\underline{\lambda}}=[\lambda_{ij}]_{ij} \in \mathbb{A}_R^{tm}$, $\overline{\underline{\underline{\lambda}}} \in U$ vary in U and $\overline{\underline{\underline{\lambda}}}$ is the image of $\underline{\underline{\lambda}}$ in $\mathbb{A}_k^{tm}$. An ideal J is called a general minimal reduction of I if J is a minimal reduction of I generated by $\ell(I)$ general elements.

The next two Theorems show that general minimal $*$-reductions exist.

THEOREM 4·4. *Let R be an excellent normal local ring of characteristic $p>0$ with infinite perfect residue field. Let I be an ideal with $*d_2(I)=1$. Then any ideal generated by $\ell^*(I)$ general elements of I is a minimal $*$-reduction of I.*

Proof. Let $\ell^*(I)=s$. Then there exists $*$-independent elements $f_1, \ldots, f_s \in I$ such that $I^*=(f_1, \ldots, f_s)^*$. Let $J=(f_1, \ldots, f_s)$. Hence J is a minimal $*$-reduction of I. By [**7**, lemma 2·2] we know that this generating set of J can be extended to a minimal generating set of I. In other words, $I=(f_1, \cdots, f_s, f_{s+1})$. By [**16**, theorem 2·1] we have that $J^*=J+J^{*sp}$. Also by [**7**, lemma 3·4] since $J \subset I$ and $J^*=I^*$ then $J^{*sp}=I^{*sp}$. Therefore $I^*=J+I^{*sp}$ and f_{s+1} can be chosen such that $f_{s+1} \in I^{*sp}$.

Let $T = R[X_{ij}]$ where $1 \leqslant i \leqslant s$ and $1 \leqslant j \leqslant s+1$. Let $a_i = \sum_{j=1}^{s+1} X_{ij} f_j$ for $1 \leqslant i \leqslant s$ and consider the T–ideal $\widetilde{J} = (a_1, \ldots, a_s)$. Write $\underline{\underline{X}}$ for $[X_{ij}]_{ij}$.

Consider the R–homomorphism $\pi_{\underline{\underline{\lambda}}} : T \to R$ that sends $\underline{\underline{X}}$ to $\underline{\underline{\lambda}}$, where $\underline{\underline{\lambda}} \in \mathbb{A}_R^{s(s+1)}$. Notice that for $\underline{\underline{\lambda_0}} = [\delta_{ij}]$ one has $\pi_{\underline{\underline{\lambda_0}}}(\widetilde{J}) = J$.

Let $\mathfrak{m}$ denote the maximal ideal of R and $k = R/\mathfrak{m}$ the residue field of R. We need to find a dense open set $U \subset \mathbb{A}_k^{s(s+1)}$, such that $\pi_{\underline{\underline{\lambda}}}(\widetilde{J})$ is also a $*$-reduction for all $\overline{\underline{\underline{\lambda}}} \in U$. Let $\overline{\lambda_{ij}}$ be the image of λ_{ij} in k. Then the generators of the k-vector space $\pi_{\underline{\underline{\lambda}}}(\widetilde{J})/\mathfrak{m}\pi_{\underline{\underline{\lambda}}}(\widetilde{J})$ are $\overline{a_i} = \sum_{j=1}^{s+1} \overline{\lambda_{ij}} f_j$.

Define $L(\underline{\underline{\lambda}}) = [\overline{\lambda_{ij}}]_{ij}$ to be the matrix defined by the coefficients of the $\overline{a_i}$ for $i = 1, \ldots, s$. Then $L(\underline{\underline{\lambda}})$ is a $s \times (s+1)$ matrix with coefficients in k. Suppose $L_s(\underline{\underline{\lambda}})$ is the $s \times s$ submatrix of $L(\underline{\underline{\lambda}})$ obtained by omitting the last column. We define the open set $U \subset \mathbb{A}_k^{s(s+1)}$ to be set of $L(\underline{\underline{\lambda}})$'s satisfying $\det(L_s(\underline{\underline{\lambda}})) \neq 0$. Since $\{L(\underline{\underline{\lambda}}) \mid \det(L_s(\underline{\underline{\lambda}})) = 0\}$ is closed, then clearly U is open. Also $L(\underline{\underline{\lambda_0}}) \in U$ and thus U is not empty. Therefore U is an open dense set (see for example [**19**, lemma 2·9]).

Since for any $\overline{\underline{\underline{\lambda}}} \in U$, $\pi_{\underline{\underline{\lambda}}}(\widetilde{J})$ is a general reduction with $\det(L_s) \neq 0$, then $V = \pi_{\underline{\underline{\lambda}}}(\widetilde{J})/\mathfrak{m}\pi_{\underline{\underline{\lambda}}}(\widetilde{J})$ is an s-dimensional k-vector space with basis $\overline{a_1}, \ldots, \overline{a_s}$. Row reducing $L(\overline{\underline{\underline{\lambda}}})$, we obtain the following matrix:

$$\begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & \beta_1' \\ 0 & 1 & 0 & \cdots & 0 & \beta_2' \\ 0 & 0 & 1 & \cdots & 0 & \beta_3' \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & \beta_s' \end{pmatrix}$$

where the $\beta_i' \in k$. This implies that an alternate basis for V is $\{\overline{f_1} + \beta_1'\overline{f_{s+1}}, \ldots, \overline{f_s} + \beta_s'\overline{f_{s+1}}\}$. Let $J_{\text{gen}} = (f_1 + \beta_1 f_{s+1}, \ldots, f_s + \beta_s f_{s+1}) = \pi_{\underline{\underline{\lambda}}}(\widetilde{J})$, where β_i is a preimage of β_i' in R.

Case 1. Suppose that for all $1 \leqslant i \leqslant s$ we have that $\beta_i \in \mathfrak{m}$. Let $K = J_{\text{gen}} + \mathfrak{m}I$. Then we claim that $K = J + \mathfrak{m}I$. To see this it is enough to check the inclusions for the generators of the ideals. Let α be a generator of K. Then we can write $\alpha = f_i + \beta_i f_{s+1} + \delta$, where $\delta \in \mathfrak{m}I$. But as $\beta_i \in \mathfrak{m}$ and $f_i \in J$ then $\alpha \in J + \mathfrak{m}I$. Now let α' be a generator of $J + \mathfrak{m}I$. Then $\alpha' = f_i + \delta'$, where $\delta' \in \mathfrak{m}I$. Since $\beta_i \in \mathfrak{m}$ then $\delta' - \beta_i f_{s+1} \in \mathfrak{m}I$. Hence $\alpha' = f_i + \beta_i f_{s+1} + (\delta' - \beta_i f_{s+1}) \in J_{\text{gen}} + \mathfrak{m}I = K$.

Next we claim that $(J + \mathfrak{m}I)^* = J^*$. Notice that $J \subset J + \mathfrak{m}I \subset I$. Taking the tight closure we obtain $J^* \subset (J + \mathfrak{m}I)^* \subset I^* = J^*$. Thus, $(J + \mathfrak{m}I)^* = J^*$. Overall we have the following inclusions:

$$J_{\text{gen}} \subset I \subset (J + \mathfrak{m}I)^* = (J_{\text{gen}} + \mathfrak{m}I)^*$$

Now by [**7**, proposition 2·1] we have that $I \subset J_{\text{gen}}^*$.

Case 2. Suppose that $\beta_i \notin \mathfrak{m}$ for some i. Then without loss of generality we may assume that $i = s$ and $\beta_s = 1$. Then $J_{\text{gen}} = (f_1 + \beta_1 f_{s+1}, \ldots, f_s + f_{s+1})$. Hence $f_1 - \beta_1 f_s \in J_{\text{gen}}$. Let $f_1' = f_1 - \beta_1 f_s$ and replace f_1 with f_1'. Continuing this way we may assume that $J_{\text{gen}} = (f_1, \ldots, f_{s-1}, f_s + f_{s+1})$. Suppose that $f_{s+1} \notin (f_1, \ldots, f_{s-1}, f_s + f_{s+1})^*$.

Since $f_{s+1}\in I\subset J^*$, then we may take $c\in R^0$ such that $cf_{s+1}^q\in J^{[q]}$ for every $q=p^e\geqslant 0$. Hence $cf_{s+1}^q=\sum_{i=1}^{s} r_{iq}f_i^q$. Then

$$cf_{s+1}^q+r_{sq}f_{s+1}^q=\sum_{i=1}^{s} r_{iq}f_i^q+r_{sq}f_{s+1}^q=\sum_{i=1}^{s-1} r_{iq}f_i^q+r_{sq}(f_s^q+f_{s+1}^q).$$

Let $c_q=c+r_{sq}$. Then $c_qf_{s+1}^q=\sum_{i=1}^{s-1} r_{iq}f_i^q+r_{sq}(f_s^q+f_{s+1}^q)$ and in particular $c_qf_{s+1}^q\in(f_1,\ldots,f_{s-1},f_s+f_{s+1})^{[q]}$. Therefore by Proposition 2·4 there is a q_0, such that $c_q\in\mathfrak{m}^{q/q_0}$ for all $q\geqslant q_0$. Also there is some q_1, such that $c\notin\mathfrak{m}^{q_1}$. Hence for all $q\geqslant q_1$, we have $r_{sq}=c_q-c\notin\mathfrak{m}^{q_1}$.

Notice that $r_{sq}f_s^q=cf_{s+1}^q-\sum_{i=1}^{s-1} r_{iq}f_i^q$. Since $r_{sq}\notin\mathfrak{m}^{q_1}$ then $f_s\in(f_1,\ldots,f_{s-1},f_{s+1})^*$ by Proposition 2·4. Therefore $(f_1,\ldots,f_{s-1},f_{s+1})^*=I^*$ and thus $(f_1,\ldots,f_{s-1},f_{s+1})$ is a minimal $*$-reduction of I. However, since $f_{s+1}\in I^{*sp}$ then $(f_1,\ldots,f_{s-1},f_{s+1})$ is not a minimal $*$-reduction of I, according to [**31**, proposition 1·12(b)], which is a contradiction. Therefore $f_{s+1}\in J_{\text{gen}}^*$ and thus $J_{\text{gen}}^*=I^*$.

We are able to generalize Theorem 4·4 and relax the condition on $*d_2(I)$.

THEOREM 4·5. *Let R be an excellent normal local ring of characteristic $p>0$ with infinite perfect residue field. Let I be an ideal. Then any ideal generated by $\ell^*(I)$ general elements of I is a minimal $*$-reduction of I.*

Proof. Let $\ell^*(I)=s$. Then there exist $*$-independent elements $f_1,\ldots,f_s\in I$ such that $I^*=(f_1,\ldots,f_s)^*$. Let $J=(f_1,\ldots,f_s)$. Hence J is a minimal $*$-reduction of I. By [**7**, lemma 2·2] we know that any generating set of J can be extended to a minimal generating set of I. In other words, there exist $f_{s+1},\ldots,f_{s+n}\in I$ such that $I=(f_1,\cdots,f_s,f_{s+1},\ldots,f_{s+n})$, where $n=*\,d_2(I)$. By [**16**, theorem 2·1] we have that $J^*=J+J^{*sp}$. Also by [**7**, lemma 3·4] since $J\subset I$ and $J^*=I^*$ then $J^{*sp}=I^{*sp}$. Therefore $I^*=J+I^{*sp}$ and thus $f_{s+1},\ldots,f_{s+n}$ can be chosen such that $f_{s+1},\ldots,f_{s+n}\in I^{*sp}$.

Let $\mathfrak{m}$ denote the maximal ideal of R and $k=R/\mathfrak{m}$ be the residue field of R. As above we form an ideal generated by general elements and we may assume that

$$J_{\text{gen}}=(f_1+\beta_{11}f_{s+1}+\cdots+\beta_{1n}f_{s+n},\ldots,f_s+\beta_{s1}f_{s+1}+\cdots+\beta_{sn}f_{s+n}),$$

where $\beta_{ij}\in R$.

Case 1. Suppose that for all $1\leqslant i\leqslant s$ and for all $1\leqslant j\leqslant n$ we have that $\beta_{ij}\in\mathfrak{m}$. Let $K=J_{\text{gen}}+\mathfrak{m}I$. Then we claim that $K=J+\mathfrak{m}I$. Let α be a generator of K. The we can write $\alpha=f_i+\beta_{i1}f_{s+1}+\cdots+\beta_{in}f_{s+n}+\delta$, where $\delta\in\mathfrak{m}I$. But as $\beta_{ij}\in\mathfrak{m}$ and $f_i\in J$ then $\alpha\in J+\mathfrak{m}I$. Now let α' be a generator of $J+\mathfrak{m}I$. Then $\alpha'=f_i+\delta'$, where $\delta'\in\mathfrak{m}I$. Since $\beta_{ij}\in\mathfrak{m}$ for all $1\leqslant i\leqslant s$ and for all $1\leqslant j\leqslant n$ then $\delta'-(\beta_{i1}f_{s+1}+\cdots+\beta_{in}f_{s+n})\in\mathfrak{m}I$. Hence $\alpha'=f_i+\beta_{i1}f_{s+1}+\cdots+\beta_{in}f_{s+n}+(\delta'-(\beta_{i1}f_{s+1}+\cdots+\beta_{in}f_{s+n}))\in J_{\text{gen}}+\mathfrak{m}I=K$.

Next we claim that $(J+\mathfrak{m}I)^*=J^*$. Notice that $J\subset J+\mathfrak{m}I\subset I$. Taking the tight closure we obtain $J^*\subset(J+\mathfrak{m}I)^*\subset I^*=J^*$. Thus $(J+\mathfrak{m}I)^*=J^*$. Overall we have the following inclusions:

$$J_{\text{gen}}\subset I\subset(J+\mathfrak{m}I)^*=(J_{\text{gen}}+\mathfrak{m}I)^*.$$

Now by [**7**, proposition 2·1] we have that $I\subset J_{\text{gen}}^*$.

Case 2. Suppose $\beta_{ij}\notin\mathfrak{m}$ for some $1\leqslant i\leqslant s$ and $1\leqslant j\leqslant n$. Without loss of generality we may assume that $i=s$ and $j=n$ and that $\beta_{sn}=1$. Hence

$$J_{\text{gen}}=(f_1+\beta_{11}f_{s+1}+\cdots+\beta_{1n}f_{s+n},\ldots,f_s+\beta_{s1}f_{s+1}+\cdots+f_{s+n}).$$

We claim that $I^* = J^*_{gen}$. We will proceed by induction on $n = *\, d_2(I) = \mu(I) - \ell^*(I)$. If $n = 0$ there is nothing to show and if $n = 1$ then Theorem 4·4 gives the result. So we assume that $n > 1$ and the result holds for any ideal I' with $*d_2(I') = n - 1$.

Let $g_i = f_i + \beta_{i1} f_{s+1} + \cdots + \beta_{in} f_{s+n}$. Then

$$g_i - \beta_{in} g_s = (f_i - \beta_{in} f_s) + \sum_{j=1}^{n-1} (\beta_{ij} - \beta_{in}\beta_{sj}) f_{s+j} \in J_{\mathrm{gen}}.$$

Notice that $f_i' = f_i - \beta_{in} f_s \in J$ and let $\beta'_{ij} = \beta_{ij} - \beta_{in}\beta_{sj}$. Therefore, we can replace f_i with f_i' and β_{ij} with β'_{ij} to assume that $J_{\mathrm{gen}} = (h_1, \ldots, h_{s-1}, h_s + f_{s+n})$, where

$$h_i = f_i + \beta_{i1} f_{s+1} + \cdots + \beta_{i(n-1)} f_{s+n-1}.$$

Let $L = (h_1, \ldots, h_s)$ and $J_1 = (f_1, \ldots, f_s, f_{s+1}, \ldots, f_{s+n-1})$.

Since g_i is a general element for all $1 \leqslant i \leqslant s$ then there exists $U \subset \mathbb{A}_k^{s(s+n)}$ a dense open set such that the image of $\beta_i = [0, \ldots, 0, 1, 0, \ldots, \beta_{i1}, \ldots, \beta_{i(n-1)}, \beta_{in}]$ varies in U. Consider the natural projection $\pi : \mathbb{A}_k^{s(s+n)} \to \mathbb{A}_k^{s(s+n-1)}$ such that $\pi((\underline{a_1}, \ldots, \underline{a_{s+n-1}}, \underline{a_{s+n}})) = (\underline{a_1}, \ldots, \underline{a_{s+n-1}})$ for $\underline{a_i} \in \mathbb{A}_k^s$. Let $W = \pi(U)$. As U is dense and open then $U \neq \varnothing$ and thus $W \neq \varnothing$ and W is also open, since π is an open map. Therefore W is a dense open set. As β_i is allowed to vary in U then $\pi(\beta_i)$ varies in W and thus h_i is also a general element of J_1.

Notice that $J \subset J_1 \subset I$ and thus $J^* = J_1^*$. Hence $\ell^*(J_1) = s$. Therefore $*d_2(J_1) \leqslant n - 1$ and by our inductive hypothesis L is a minimal $*$-reduction of J_1 and thus of I. Hence $L^* = J_1^* = J^* = I^*$. We are claiming that $J^*_{\mathrm{gen}} = L^* = I^*$. It is enough to show that $f_{s+n} \in J^*_{\mathrm{gen}}$. Suppose that $f_{s+n} \notin J^*_{\mathrm{gen}}$. Then as in the proof of Theorem 4·4 we obtain that $h_s \in (h_1, \ldots, h_{s-1}, f_{s+n})^*$, which implies that $L^* = (h_1, \ldots, h_{s-1}, f_{s+n})^* = I^*$. However, since $f_{s+n} \in I^{*sp}$ then $(h_1, \ldots, h_{s-1}, f_{s+n})$ is not a minimal $*$-reduction of I, by [**31**, proposition 1·12(b)], which is a contradiction. Hence $f_{s+n} \in J^*_{\mathrm{gen}}$ and $J^*_{\mathrm{gen}} = L^* = I^*$.

COROLLARY 4·6. *Let $(R, \mathfrak{m})$ be a Gorenstein normal local isolated singularity of characteristic $p > 0$ with infinite perfect residue field. Suppose that the test ideal of R is $\mathfrak{m}$. Let $I = J^*$ where J is a parameter ideal minimally generated by s elements. Then any ideal generated by s general elements of I is a minimal $*$-reduction of I.*

Proof. Suppose $J = (f_1, \ldots, f_s)$. Then J is a minimal $*$-reduction of $I = J^* = (J : \mathfrak{m})$, where the last equality is obtained by Theorem 2·2. Since R is Gorenstein then the socle $(J : \mathfrak{m})/J$ is a one dimensional vector space. Hence $I = (f_1, \ldots, f_s, f_{s+1})$, where $f_{s+1} \notin J$. Therefore $\mu(I) = s + 1$ and $*d_2(I) = 1$. Thus by Theorem 4·4, any ideal generated by s general elements is a minimal $*$-reduction of I.

COROLLARY 4·7. *Let $(R, \mathfrak{m})$ be a Gorenstein normal local isolated singularity of characteristic $p > 0$ with infinite perfect residue field. Suppose that the test ideal of R is $\mathfrak{m}$. Let $I = J^*$ where J is generated by a system of parameters. Then* $\mathrm{core}(I) = *\text{-}\mathrm{core}(I)$.

Proof. Since I is $\mathfrak{m}$-primary then $\ell(I) = \ell^*(I) = d$, where $d = \dim R$. By Corollary 4·6 any ideal generated by d general elements is a general minimal $*$-reduction. Notice that these general minimal $*$-reductions are also general minimal reductions of I, since $\ell(I) = d$.

Also, since I is $\mathfrak{m}$-primary then by Theorem 2·6 ([**4**, theorem 4·5]) we have that $\mathrm{core}(I)$ is a finite intersection of general minimal reductions. Since each general minimal reduction is also a minimal $*$-reduction then $*\text{-}\mathrm{core}(I) \subset \mathrm{core}(I)$. On the other hand $\mathrm{core}(I) \subset *\text{-}\mathrm{core}(I)$, by Corollary 3·5.

THEOREM 4·8. *Let R be a Cohen–Macaulay normal local domain of characteristic $p > 0$ with infinite perfect residue field. Let I be an ideal with $\ell^*(I) = \ell(I) = s$. We further assume that I satisfies G_s and is weakly $(s-1)$-residually S_2. Then* $\mathrm{core}(I) = *\text{-}\mathrm{core}(I)$.

Proof. We know that $\mathrm{core}(I) \subset *\text{-}\mathrm{core}(I)$ by Corollary 3·5. According to Theorem 2·6 the core is a finite intersection of general minimal reductions. Since every general minimal reduction is a minimal $*$-reduction by Theorem 4·5, we obtain the opposite inclusion.

5. *The $*$-core in complete one dimensional semigroup rings*

In Proposition 4·1, we saw that the core and the $*$-core agree for all ideals in a one dimensional domain of characteristic $p > 0$ with infinite residue field. In Huneke and Swanson's paper [**14**], one of the first questions that they ask is: if I is integrally closed, is $\mathrm{core}(I)$ integrally closed? They settle this question in the setting of a two-dimensional regular local ring. Corso, Polini and Ulrich in [**5**, theorem 2·11] showed that if R is a Cohen–Macaulay normal local ring with infinite residue field then $\mathrm{core}(I)$ is integrally closed, when I is a normal ideal of positive height, universally weakly $(\ell - 1)$-residually S_2 and satisfies G_ℓ and $AN^-_{\ell-1}$, where $\ell = \ell(I)$. A related question is: if I is tightly closed, is $*$-core(I) tightly closed? We will consider this question now for complete one-dimensional semigroup rings with test ideal equal to the maximal ideal. The second author showed the following:

THEOREM 5·1 ([**27**]). *Let R be a one-dimensional domain. The test ideal of R is equal to the conductor, i.e. $\tau = \mathfrak{c} = \{c \in R \mid \phi(1) = c, \phi \in \mathrm{Hom}_R(\overline{R}, R)\}$, where $\overline{R}$ denotes the integral closure of R.*

Note that in a one-dimensional local semigroup ring, the semigroup is a sub-semigroup of $\mathbb{N}_0$. For each sub-semigroup S of $\mathbb{N}_0$, there is a smallest m such that for all $i \geqslant m$, $i \in S$. The conductor of such a one dimensional semigroup ring is $\mathfrak{c} = < t^m, t^{m+1}, t^{m+2}, \ldots >$, [**6**, exercise 21·11]. Hence the test ideal in a one-dimensional semigroup ring is the maximal ideal, if the conductor is the maximal ideal. This can only happen if the semigroup has the form $\{n + i \mid i \geqslant 0\}$ for some $n \geqslant 0$. Hence if R is complete the ring is of the form $R = k[[t^n, t^{n+1}, \ldots, t^{2n-1}]]$, where k is a field. As in [**28**, proposition 4·1], we will show that the principal ideals of R are of a given form:

PROPOSITION 5·2. *Let $R = k[[t^n, t^{n+1}, \ldots, t^{2n-1}]]$ be a one-dimensional local semigroup ring and k a field. Each nonzero nonunit principal ideal of R can be expressed in the form*

$$(t^m + a_1 t^{m+1} + \cdots + a_{n-1} t^{m+n-1}),$$

where $a_i \in k$ and $m \geqslant n$.

Proof. Suppose $0 \neq f \in R$. Thus, after multiplying by a nonzero element of k we may assume that $f = t^m + a_1 t^{m+1} + a_2 t^{m+2} + \cdots$ for some $a_{ij} \in k$ and for some $m \geqslant n$. We will show that $t^r \in (f)$ for $r \geqslant m + n$. Hence $t^m + a_1 t^{m+1} + \cdots + a_{n-1} t^{m+n-1} \in (f)$. Similarly, for $r \geqslant m + n$ we obtain $t^r \in (t^m + a_1 t^{m+1} + \cdots + a_{n-1} t^{m+n-1})$. Hence $f \in (t^m + a_1 t^{m+1} + \cdots + a_{n-1} t^{m+n-1})$.

Let $g \in k[[t]]$. Note that $t^{r-m} g \in R$. Therefore if g is a unit in $k[[t]]$, then $t^{r-m} g^{-1} \in R$ also. In $k[[t]]$ we have $f = t^m(1 + a_1 t + a_2 t^2 + \cdots) = t^m g$, for some unit $g \in k[[t]]$. Also notice that $t^{r-m} g^{-1} f = t^r$.

Similarly $t^r \in (t^m + a_1t^{m+1} + \cdots + a_{n-1}t^{m+n-1})$. Since

$$f - (t^m + a_1t^{m+1} + \cdots + a_{n-1}t^{m+n-1}) = a_nt^{2n} + a_{n+1}t^{2n+1} + \cdots$$

is an element of $(f)\bigcap(t^m + a_1t^{m+1} + \cdots + a_{n-1}t^{m+n-1})$, it follows that

$$(t^m + a_1t^{m+1} + \cdots + a_{n-1}t^{m+n-1}) = (f).$$

Therefore all principal ideals of R are of the form $(t^m + a_1t^{m+1} + \cdots + a_{n-1}t^{m+n-1})$.

PROPOSITION 5·3. *Let $R = k[[t^n, t^{n+1}, \ldots, t^{2n-1}]]$ be a one-dimensional local semigroup ring and k be an infinite field of characteristic $p > 0$. Any tightly closed ideal in R is of the form $(t^m, t^{m+1}, \ldots, t^{m+n-1})$ for some $m \geqslant n$.*

Proof. Suppose I is a tightly closed ideal in R. Since R is a one-dimensional local domain, there is a principal ideal $(f) \in I$, with $(f)^* = I$. By Proposition 5·2, $(f) = (t^m + a_1t^{m+1} + \cdots + a_{n-1}t^{m+n-1})$, for some $m \geqslant n$ and $a_i \in k$. According to [**12**, example 1·6·2], $I^* = \overline{I} = \overline{(x)}$ for some $x \in I$. Moreover, by [**27**, theorem 3·8] it follows that $\overline{(x)} = (x)\overline{R} \bigcap R$. Hence

$$(f)^* = \overline{(f)} = (f)k[[t]] \bigcap R = (t^m, t^{m+1}, \ldots, t^{m+n-1}).$$

PROPOSITION 5·4. *Let $R = k[[t^n, t^{n+1}, \ldots, t^{2n-1}]]$ be a one-dimensional local semigroup ring and k be an infinite field of characteristic $p > 0$. Let I be a tightly closed ideal. Then $*$-core(I) is also tightly closed.*

Proof. If $I = (0)$, then clearly $*$-core$(I) = (0)$ and thus the assertion is clear. Since R is a one-dimensional domain then core$(I) = *$-core(I) by Proposition 4·2. If I is basic then I is also $*$-basic and again the assertion is clear. So suppose I is not basic, nonzero and tightly closed. Then $I = (t^m, t^{m+1}, \ldots, t^{m+n-1})$ for some $m \geqslant n$, by Proposition 5·3. Since I is non-zero then I is $\mathfrak{m}$-primary, where $\mathfrak{m}$ is the maximal ideal of R. Hence by Theorem 2·6 we have that core$(I) = \bigcap_{i=1}^{s}(f_i)$, for some positive integer s and (f_i) general minimal reductions of I for all $1 \leqslant i \leqslant s$. Let (f_i) be such a general minimal reduction. Then

$$(f_i) = (t^m + a_{i1}t^{m+1} + \cdots + a_{i(n-1)}t^{m+n-1})$$

for some $a_{ij} \in k$, since f_i is a general element in I. As in the proof of Proposition 5·2, we see that $t^r \in (t^m + a_1t^{m+1} + \cdots + a_{n-1}t^{m+n-1})$ for all $r \geqslant m + n$. Hence $(t^{m+n}, t^{m+n+1}, \ldots, t^{m+2n-1}) \subset (f_i)$ for all i and thus $(t^{m+n}, t^{m+n+1}, \ldots, t^{m+2n-1}) \subset *$-core$(I)$.

On the other hand let $g \in *$-core(I). Hence $g \in \bigcap_{i=1}^{s}(f_i)$. It is clear that $(g) \neq (f_i)$ for some i. Then $g = a(t^m + a_{i1}t^{m+1} + \cdots + a_{i(n-1)}t^{m+n-1})$ for some $a \in R$ and $a_{ij} \in k$. If a is a unit then $(g) = (f_i)$, which is a contradiction. Hence we may assume that a is not a unit. Thus

$$a = \beta_1t^n + \beta_2t^{n+1} + \cdots \text{ and}$$
$$g = \gamma_0t^{m+n} + \cdots + \gamma_{n-1}t^{m+2n-1} + t^n(\gamma_nt^{m+n} + \cdots + \gamma_{2n-1}t^{m+2n-1}) + \cdots .$$

Therefore $g \in (t^{m+n}, t^{m+n+1}, \ldots, t^{m+2n-1})$ and thus

$$*\text{-core}(I) \subset (t^{m+n}, t^{m+n+1}, \ldots, t^{m+2n-1}).$$

Finally notice that $*$-core$(I) = (t^{m+n}, t^{m+n+1}, \ldots, t^{m+2n-1})$ is a tightly closed ideal.

Note that by Proposition 4·1 in a one-dimensional domain with infinite residue field we have $\mathrm{core}(I) = *\text{-core}(I)$ and the tight closure of an ideal agrees with the integral closure. Thus we obtain:

COROLLARY 5·5. *Let $R = k[[t^n, t^{n+1}, \ldots, t^{2n-1}]]$ be a one-dimensional local semigroup ring and k be an infinite field of characteristic $p > 0$. Let I be an integrally closed ideal. Then* $\mathrm{core}(I)$ *is integrally closed.*

Remark 5·6. As mentioned above the question of whether the core of an integrally closed ideal is also integrally closed was first addressed by Huneke and Swanson [**13**]. They answer this question positively when the ring is a 2–dimensional regular ring [**13**, corollary 3·12]. This question was also addressed by several other authors later (see [**5**, theorem 2·11, corollary 3·7], [**23**, corollary 4·6], and [**17**, proposition 5·5·3]).

We note here that Corollary 5·5 is not covered by any of the results mentioned above. In [**5**, corollary 3·7] and [**23**, corollary 4·6] it is required that the ring R is Gorenstein. The ring $R = k[[t^n, t^{n+1}, \ldots, t^{2n-1}]]$ with k an infinite field of characteristic $p > 0$ is not Gorenstein unless $n = 2$. In [**4**, theorem 2·11] the Gorenstein condition can be relaxed to Cohen–Macaulay rings, but in addition the Rees algebra of I and I are assumed to be normal and $J : I$ is independent of J for every minimal reduction J of I. Notice that the ideal I in Corollary 5·5 is normal and $J : I = \tau = \mathfrak{m}$ is independent of the minimal reduction J of I. However, the Rees algebra of I is not normal, since R is not normal. Finally, in [**17**, proposition 5·5·3] it is assumed that the ring R is Cohen–Macaulay, R contains the rational numbers and the Rees algebra of I is Cohen–Macaulay whereas the ring in Corollary 5·5 need not contain the rational numbers.

6. *Examples*

Since the tight closure of an ideal is much closer to the ideal than the integral closure we expected to find examples of ideals I where the $\mathrm{core}(I) \subsetneq *\text{-core}(I)$. The following example gives a family of rings where $\mathrm{core}(\mathfrak{m}^2) \subsetneq *\text{-core}(\mathfrak{m}^2)$.

Example 6·1. Let $R = \mathbb{Z}/p\mathbb{Z}(u, v, w)[[x, y, z]]/(ux^p + vy^p + wz^p)$, where p is prime. Then R is a normal domain [**7**]. In [**29**] Vraciu and the second author computed the test ideal of R to be $\mathfrak{m}^{p-1}$, where $\mathfrak{m}$ is the maximal ideal of R. Let $k = \mathbb{Z}/p\mathbb{Z}(u, v, w)$. Notice that since $\ell(\mathfrak{m}^2) = 2$ then $\ell^*(\mathfrak{m}^2)$ is either 2 or 3.

We begin by showing that $\ell^*(\mathfrak{m}^2) = 3$, regardless of the characteristic p. We claim that $J = (y^2, yz, z^2)$ is a minimal $*$-reduction of $\mathfrak{m}^2$. To establish this we must show that y^2, yz, z^2 are $*$-independent elements and that $J^* = \mathfrak{m}^2$.

We note that y^2, z^2 is a system of parameters and therefore by Theorem 2·2 we have $(y^2, z^2)^* = (y^2, z^2) : \mathfrak{m}^{p-1}$. Hence,

$$\begin{aligned}(y^2, z^2)^* &= (y^2, z^2) : \mathfrak{m}^{p-1} = (y^2, z^2, x^{p-1}yz) : \mathfrak{m}^{p-2} \\ &= (y^2, z^2, x^{p-1}y, x^{p-1}z, x^{p-2}yz) : \mathfrak{m}^{p-3} = \cdots = (y^2, z^2) + \mathfrak{m}^3.\end{aligned}$$

In particular, this shows that $yz \notin (y^2, z^2)^*$. It remains to establish that $z^2 \notin (y^2, yz)^*$ and $y^2 \notin (yz, z^2)^*$. Notice that $(y^2, yz)^* \subset (y^2, yz) : \mathfrak{m}^{p-1}$, by the definition of the test ideal. As above,

$$(y^2, yz) : \mathfrak{m}^{p-1} = (y^2, yz, x^{p-1}y) : \mathfrak{m}^{p-2} = (y^2, z^2, x^{p-2}y) : \mathfrak{m}^{p-3} = \cdots = (y^2, yz, xy).$$

One then observes that since $z^2 \notin (y^2, yz, xy)$ then $z^2 \notin (y^2, yz)^*$. Similarly $y^2 \notin (yz, z^2)^*$. Therefore y^2, yz, z^2 are $*$-independent elements.

Next we must show that $J^* = \mathfrak{m}^2$. The calculations depend on the characteristic p and thus we separate the computations.

For $p = 2$ notice that $x^2 = (v/u)y^2 + (w/u)z^2 \in J$ and

$$(xz)^2 = ((v/u)y^2 + (w/u)z^2)z^2 = (v/u)(yz)^2 + (w/u)(z^2)^2.$$

Hence $xz \in J^F \subset J^*$. Similarly $xy \in J^*$ and thus $\mathfrak{m}^2 = J^*$. Therefore J is indeed a minimal $*$-reduction of $\mathfrak{m}^2$.

For $p \geqslant 3$ notice that

$$(x^2)^p = (x^p)^2 = ((v/u)y^p + (w/u)z^p)^2 = (v^2/u^2)(y^2)^p + 2(vw/u^2)(yz)^p + (w^2/u^2)(z^2)^p \in J^F$$

and $(xz)^p = ((v/u)y^p + (w/u)z^p)z^p$. Thus $xz \in J^F \subset J^*$. Similarly $xy \in J^*$ and thus $J^* = \mathfrak{m}^2$. Therefore J is again a minimal $*$-reduction of $\mathfrak{m}^2$ and thus $l^*(\mathfrak{m}^2) = 3$ for any characteristic.

Next we continue with the computations of $*$-core$(\mathfrak{m}^2)$ and core$(\mathfrak{m}^2)$. Once again these depend on the characteristic p and thus we separate the computations.

For $p = 2$, we compute the $*$-core of $\mathfrak{m}^2$ in the following manner: Recall that $J = (y^2, yz, z^2)$ is a minimal $*$-reduction of $\mathfrak{m}^2$. In addition we note that (x^2, xy, y^2), (x^2, xz, z^2), (y^2, yz, z^2), and (yz, xz, xy) are all minimal $*$-reductions of $\mathfrak{m}^2$. Hence

$$\begin{aligned} *\text{-core}(\mathfrak{m}^2) &\subset (x^2, xy, y^2) \bigcap (x^2, xz, z^2) \bigcap (y^2, yz, z^2) \bigcap (yz, xz, xy) \\ &= (x^2, y^2, z^2, xyz) \bigcap (yz, xz, xy) = \mathfrak{m}^3. \end{aligned}$$

Note that $\mathfrak{m}^3 = \mathfrak{m}J^* \subset J$ for all J minimal $*$-reductions of $\mathfrak{m}^2$, since $\mathfrak{m}$ is the test ideal. Hence $*$-core$(\mathfrak{m}^2) = \mathfrak{m}^3$.

For $p \geqslant 3$ we estimate the $*$-core of $\mathfrak{m}^2$. Again, $J = (y^2, yz, z^2)$ is a minimal $*$-reduction of $\mathfrak{m}^2$ and similarly (x^2, xy, y^2), (x^2, xz, z^2), (y^2, yz, z^2), and (yz, xz, xy) are all minimal $*$-reductions of $\mathfrak{m}^2$. As the test ideal is $\mathfrak{m}^{p-1}$, we see that $\mathfrak{m}^{p-1}J^* = \mathfrak{m}^{p+1} \subset J$ for all minimal $*$-reductions J of $\mathfrak{m}^2$. Therefore

$$\begin{aligned} *\text{-core}(\mathfrak{m}^2) &\subset (x^2, xy, y^2) \bigcap (x^2, xz, z^2) \bigcap (y^2, yz, z^2) \bigcap (yz, xz, xy) \\ &= m^{p+1} + (xyz, x^2y^2, x^2z^2, y^2z^2). \end{aligned}$$

Hence $\mathfrak{m}^{p+1} \subset *$-core$(\mathfrak{m}^2) \subset \mathfrak{m}^{p+1} + (xyz, x^2y^2, x^2z^2, y^2z^2)$. We remark here that we have not been able to establish a closed formula for $*$-core$(\mathfrak{m}^2)$ for $p \geqslant 3$, but we will show that the above inclusions are enough to show that core$(\mathfrak{m}^2) \subsetneq *$-core$(\mathfrak{m}^2)$.

Last we compute core$(\mathfrak{m}^2)$. Recall that $\ell(\mathfrak{m}^2) = 2$ and notice that $H = (x^2, yz)$ is a minimal reduction of $\mathfrak{m}^2$ in any characteristic.

For $p = 2$ the reduction number of $\mathfrak{m}^2$ with respect to H is 1. Since char $k = 2 > 1$ then we may use the formula for the core as in [**23**, theorem 4·5]. Hence core$(\mathfrak{m}^2) = H^2 : \mathfrak{m}^2 = \mathfrak{m}^4$, where the last equality follows from calculations using the computer algebra program Macaulay 2 [**21**]. Therefore $\mathfrak{m}^4 = $ core$(\mathfrak{m}^2) \subsetneq *$-core$(\mathfrak{m}^2) = \mathfrak{m}^3$.

For $p = 3$ the reduction number of $\mathfrak{m}^2$ with respect to H is 2. Since now char $k = 3 > 2$ we may again use the formula as in [**23**, theorem 4·5]. Thus core$(\mathfrak{m}^2) = H^3 : \mathfrak{m}^4 = \mathfrak{m}^5$, where the last equality is again obtained using the computer algebra program Macaulay 2 [**21**]. Notice that since $\mathfrak{m}^4 \subset *$-core$(\mathfrak{m}^2)$ then core$(\mathfrak{m}^2) \subsetneq *$-core$(\mathfrak{m}^2)$ again.

When the analytic spread and the $*$-spread agree, it is not necessarily the case that all reductions of an ideal are $*$-reductions. However, the following example exhibits that even so, the core and the $*$-core agree for the maximal ideal in the following ring. In some sense, the following example prompted us to prove Theorem 4·4, Theorem 4·5 and Theorem 4·8.

Example 6·2. Let $R = k[[x, y, z]]/(x^2 - y^3 - z^7)$, where the k is an infinite field and char $k > 7$. Let $\mathfrak{m} = (x, y, z)$ denote the maximal ideal of R. We observe first that $\mathfrak{m}$ is the test ideal, [**27**].

We will show that $*$-spread of $\mathfrak{m}$ is 2, $\ell(\mathfrak{m}) = 2$ and $\mathrm{core}(\mathfrak{m}) = \mathfrak{m}^2 = *\text{-core}(\mathfrak{m})$.

First note that R is a 2–dimensional Gorenstein local ring and hence $\ell(\mathfrak{m}) = 2$. Let $J = (y, z)$. Then J is a minimal reduction of $\mathfrak{m}$ with reduction number 1. Since char $k > 1$ then $\mathrm{core}(\mathfrak{m}) = J^2 : \mathfrak{m} = \mathfrak{m}^2$ by [**23**, theorem 4·5]. Notice that this does not agree with the formula in Hyry-Smith [**18**, theorem 4·1] or in Fouli–Polini–Ulrich [**8**, theorem 4·4] since $a = 42 - 21 - 14 - 6 = 1$ and $\mathrm{core}(\mathfrak{m}) \neq \mathfrak{m}^{2+a+1} = \mathfrak{m}^4$. The hypothesis that $\mathfrak{m}$ is generated by elements of degree 1 is important in their formula.

On the other hand, J is also a minimal $*$-reduction of $\mathfrak{m}$. Note that y, z form a system of parameters and by Theorem 2·2 we have that $(y, z)^* = (y, z) : \mathfrak{m} = (x, y, z) = \mathfrak{m}$. Therefore $\ell^*(\mathfrak{m}) = 2 = \ell(\mathfrak{m})$. We claim that $J_1 = (x + z, y)$ and $J_2 = (x + y, z)$ are also minimal $*$-reductions. Denote

$$p_n(x, y) = x^n + x^{n-1}y + \cdots + xy^{n-1} + y^n.$$

Note that if n is odd, $x^n + y^n = (x + y)p_{n-1}(x, -y)$. Now we can see that

$$(x + z)p_6(x, -z) + y^3 = x^7 + z^7 + y^3 = x^7 + x^2 - y^3 + y^3 = x^2(1 + x^5).$$

Since $(1 + x^5)$ is a unit in R, then $x^2 \in (x + z, y)$. Since $x(x + z) = x^2 + xz$ we also observe that $xz \in (x + z, y)$ and similarly, we see that $z^2 \in (x + z, y)$. Hence $\mathfrak{m}^2 \subset (x + z, y)$ and thus $\mathfrak{m} \subset (x + z, y) : \mathfrak{m} = (x + z, y)^* \subset \mathfrak{m}$, i.e. $(x + z, y)^* = \mathfrak{m}$. Using the same argument exchanging y and z and exchanging the powers 3 and 7, we see that indeed J_2 is a minimal $*$-reduction of $\mathfrak{m}$.

Let K be a minimal $*$-reduction of $\mathfrak{m}$. Then $\mathfrak{m} = K^*$ and $\mathfrak{m}^2 = \mathfrak{m}K^* \subset K$. Therefore $\mathfrak{m}^2 \subset *\text{-core}(\mathfrak{m})$. We can easily see that $\mathfrak{m}^2 = J \bigcap J_1 \bigcap J_2$ and hence conclude that $\mathfrak{m}^2$ is in fact $*\text{-core}(\mathfrak{m})$ and $\mathrm{core}(\mathfrak{m}) = *\text{-core}(\mathfrak{m})$.

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