

## A LOOK AT THE PRIME AND SEMIPRIME OPERATIONS OF ONE-DIMENSIONAL DOMAINS

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ABSTRACT. We continue the analysis of prime and semiprime operations over one-dimensional domains started in [Va]. We first show that there are no bounded semiprime operations on the set of fractional ideals of a one-dimensional domain. We then prove the only prime operation is the identity on the set of ideals in semigroup rings where the ideals are minimally generated by two or fewer elements. This is not likely the case in semigroup rings with ideals of three or more generators since we are able to exhibit that there is a non-identity prime operation on the set of ideals of  $k[[t^3, t^4, t^5]]$ .

### 1. INTRODUCTION

In his 1935 book *Idealtheorie* [Kr1], Krull defined an operation  $I \mapsto I'$  on the set of fractional ideals of a domain  $R$  to be a  $\iota$ -operation if it satisfies the following properties where  $I$  and  $J$  are ideals and  $b$  is a regular element:

- (1)  $I \subseteq I'$
- (2) If  $I \subseteq J$ , then  $I' \subseteq J'$ .
- (3)  $(I')' = I'$ .
- (4)  $I'J' \subseteq (IJ)'$
- (5)  $(bI)' = bI'$
- (6)  $I' + J' \subseteq (I + J)'$ .

Then a year later in [Kr2], he discussed the integral completion or  $b$ -operation in terms of  $\iota$ -operations and mentioned that he left out the properties:

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$$(7) R = R'$$

$$(8) (I' \cap J')' = I' \cap J'.$$

In fact, Sakuma [Sa] shows in 1957 that when looking at prime operations on the set of fractional ideals of a domain, properties (4), (6) and (8) are consequences of properties of (1), (2), (3), (5) and (7). In 1964, Petro [Pe] called the operations satisfying properties (1)-(4) on the set of fractional ideals, semiprime operations. After, Gilmer wrote the book, *Multiplicative Ideal Theory* [Gi] in 1972, prime operations on the set of fractional ideals came to be known as star operations. Gilmer explained in his book that  $I'$  was a common notation for integral closure and as Krull also used the notation  $I^*$  with regard to the  $v$ -operation, he preferred to call these operations star operations or  $\star$ -operations. For more on star operations and the structure on the set of star operations, one may refer to the following two articles [AA] by Dan and Dave Anderson and [AC] by Dan Anderson and Sylvia Cook.

In 1969, Kirby [Ki] defined a closure operation on the set of submodules of an  $R$ -module  $M$  for an arbitrary commutative ring to satisfy properties similar to (1), (2) and (3) above and  $(IN)' = IN'$  where  $I \subseteq R$  is an ideal and  $N$  is a  $R$ -submodule of  $M$ . This seems to be the first reference where the such operations were defined for an arbitrary commutative ring and on a set other than the set of fractional ideals over a domain. The terms prime and semiprime operation were reintroduced on the set of ideals of a commutative ring by Ratliff in his 1989 paper [Ra] on  $\Delta$ -closures of ideals. Heinzer, Ratliff and Rush [HRR] also use the term semiprime operation when referring to the basically full closure on the set of  $m$ -primary submodules of a module over a local ring  $(R, m)$ .

Certainly, when the ring  $R$  is a domain, we can determine which semiprime operations on the set of ideals of  $R$  are star operations when we extend them to the set of fractional ideals. In a recent paper [Va], the author determined all the semiprime operations on the set of ideals of a Dedekind domain and all the semiprime operations on the set of ideals of the complete cuspidal cubic. We will observe that these operations are not semiprime operations on the set of all fractional ideals; hence, they are not star operations. Also, we exhibit that over  $k[[t^3, t^4, t^5]]$ , there is a prime operation that is not the identity and it seems likely that a one-dimensional semigroup ring having ideals minimally generated by 3 or more elements, will have prime operations which are not the identity. However, if the ideals are minimally generated by two or fewer elements, then the only prime operation is the identity.

2. THERE ARE NO BOUNDED SEMIPRIME OPERATIONS ON THE SET OF FRACTIONAL IDEALS OF A ONE-DIMENSIONAL DOMAIN

Let  $R$  be an integral domain and  $K$  its field of fractions. Let

$$\mathfrak{I} = \{I \subseteq R \mid I \text{ an ideal of } R\} \text{ and } \mathfrak{F} = \{I \subseteq K \mid I \text{ a fractional ideal of } R\}.$$

Let  $M_{\mathfrak{I}} = \{f : \mathfrak{I} \rightarrow \mathfrak{I}\}$  and  $M_{\mathfrak{F}} = \{f : \mathfrak{F} \rightarrow \mathfrak{F}\}$ .  $M_{\mathfrak{I}}$  and  $M_{\mathfrak{F}}$  are clearly a monoids under composition of maps with identity, the identity map  $e : \mathfrak{I} \rightarrow \mathfrak{I}$  and function composition is associative.  $C_R$  will be the subset of  $M_{\mathfrak{I}}$  consisting of closure operations. Hence, the  $f_c$  in  $C_R$  are the set of maps satisfying the following three properties: (a)  $f_c(I) \supseteq I$ , (b)  $f_c$  preserves inclusions in  $R$ , and (c)  $f_c \circ f_c = f_c$ .  $S_R$  will be the set of semiprime operations of  $R$ , i.e.  $S_R$  are the maps in  $C_R$  which also satisfy (d)  $f_c(I)f_c(J) \subseteq f_c(IJ)$ .  $P_R$  will be the set of prime operations of  $R$ , i.e. maps in  $S_R$  which also satisfy (e)  $f_c(bI) = bf_c(I)$ .

**Definition 1.** We say a closure operation  $f_c$  is *bounded* on a commutative ring  $R$  if for every maximal ideal  $m$  of  $R$ , there is an  $m$ -primary ideal  $I$  such that for all  $m$ -primary  $J \subseteq I$ ,  $f_c(J) = I$ . If this is not the case, we will say that  $f_c$  is an *unbounded* closure operation.

The author has determined all the semiprime operations on the set of ideals of a Dedekind domain and the cuspidal cubic. In each case, there are several bounded semiprime operations. In particular, over a discrete valuation ring, all the semiprime operations excluding the identity are bounded.

**Proposition 2.1** ([Va], Proposition 3.2). *When  $(R, P)$  is a discrete valuation ring, the only semiprime operations on the set of ideals of  $R$  are the identity,  $f_m$  and  $g_m$  defined below:*

$$f_m(P^i) = \begin{cases} P^i & \text{for } 0 \leq i < m \\ P^m & \text{for } i \geq m \end{cases} \quad \text{and } f_m(0) = (0)$$

and

$$g_m(P^i) = \begin{cases} P^i & \text{for } 0 \leq i < m \\ P^m & \text{for } i \geq m \end{cases} \quad \text{and } g_m(0) = P^m.$$

Moreover, the set of semiprime operations  $S_R$  decomposes as the union  $S_R = M_0 \cup M_f$  where  $M_0 = \{e\} \cup \{f_n \mid n \geq 0\}$  and  $M_f = \{e\} \cup \{g_n \mid n \geq 0\}$  each of which are submonoids of  $M_{\mathfrak{I}}$  and  $M_f$  is a left but not a right  $M_0$ -act under composition.

We make use of the next definition in the following remark.

**Definition 2.** [Va][Definition 4.10] Let  $R$  be a one-dimensional semigroup ring defined by  $S \subseteq \mathbb{N}_0$ . Let  $f_c$  is a bounded semiprime operation and  $J$  be the unique ideal with  $f_c(I) = J$  for all  $(0) \neq I \subseteq J$  and  $n \geq 1$  be the conductor of  $S$ . Suppose  $\mathfrak{a}$  is an ideal which is incomparable to  $J$  and  $f_c(\mathfrak{a}) \supseteq J$  and  $\mathfrak{a} = \mathfrak{a}_0 \subseteq \mathfrak{a}_1 \subseteq \cdots \subseteq \mathfrak{a}_k = f_c(\mathfrak{a})$  is a composition series for  $f_c(\mathfrak{a})/\mathfrak{a}$  for  $k \geq n$  with  $\mathfrak{a}_i \supseteq J$  for all  $i > 0$ . Then we say  $f_c$  is an *exceptional semiprime operation*.

*Remark.* We would like to note before continuing, that those who study star operations exclude the zero ideal from the set of fractional ideals. If we had excluded the zero ideal from the  $M_{\mathcal{J}}$ , the set of ideals, then we observe that the semiprime operations over a DVR would be a submonoid of  $M_{\mathcal{J}}$ . Similarly, if zero were excluded, the the semiprime operations over a Dedekind domain would be a submonoid of  $M_{\mathcal{J}}$  in Proposition 3.6 of [Va]. However, even excluding zero in the case of the cuspidal cubic,  $S_R$  is not a submonoid because of the exceptional semiprime operations  $f_{n,S,T,m'}$ , see Proposition 4.11 in [Va].

If we consider these same operations on the set of fractional ideals of  $(R, P)$ , we see that all the bounded semiprime operations on  $M_{\mathcal{J}}$  are not semiprime on  $M_{\mathcal{F}}$ . Hence, the only semiprime operation on the set of fractional ideals of  $R$  is the identity.

**Proposition 2.2.** *When  $(R, P)$  is a discrete valuation ring, the only semiprime operation  $f \in M_{\mathcal{F}}$  on the fractional ideals of  $R$  is the identity.*

PROOF. First note, that there are no semiprime operations  $f \in M_{\mathcal{F}}$  on the set of fractional ideals of  $R$ , with  $R \subsetneq f(R)$  since if  $f(R) = P^i$ ,  $i < 0$ , then

$$f(R)f(R) = P^{2i} \supsetneq P^i = f(R) = f(R \cdot R).$$

The fractional ideals of  $R$  have the form  $P^i$ ,  $i \in \mathbb{Z}$  and they are totally ordered. Suppose  $f_c$  is a bounded semiprime operation, then  $f_c(P^i) = P^m$  for all  $i \geq m$ . Suppose  $i + j = m$  with  $i > m$ , then  $f_c(P^i)f_c(P^j) \subseteq f_c(P^m)$ , but since  $j = m - i < 0$ ,

$$P^{m+j} \subseteq f_c(P^i)f_c(P^j) \subseteq f_c(P^m) = P^m$$

leads to a contradiction since  $P^m \subseteq P^{m+j}$  for  $j < 0$ . □

In fact, no one-dimensional local Noetherian domain can have bounded semiprime operations on the set of fractional ideals.

**Proposition 2.3.** *Let  $(R, \mathfrak{m})$  be a local one-dimensional Noetherian domain. There are no bounded semiprime operations on the fractional ideals of  $R$ .*

PROOF. As in the above Proposition,  $f_c(R) = R$ . Suppose that  $f_c \in M_{\mathfrak{F}}$  is a semiprime operation which is bounded. For any element  $s \in \mathfrak{m}$ , and  $i \in \mathbb{Z}$ ,  $s^i R$  is a fractional ideal, and the  $s^i R$  form a chain. Suppose  $n$  is the minimal integer satisfying  $s^n R \subseteq I$  for some  $m$ -primary ideal  $I \subseteq R$  and  $f_c(s^i R) = I$  for all  $i \geq n$  but for  $i < n$ . Since  $s^i R \not\subseteq I$ , then  $f_c(s^i R) \not\subseteq I$ . Note that  $f_c(s^{i+j} R) = I$  for  $i + j = n$ . If  $j < 0$  and  $i > n$ , then as  $s^{n-j} \notin I$ , then  $s^j I \not\supseteq I$  and the following chain

$$f_c(s^i R)f_c(s^j R) = If_c(s^j R) \supseteq s^j I \not\supseteq I = f_c(s^{i+j} R)$$

implies that  $f_c$  is not a semiprime operation since  $f_c(s^j R)f_c(s^i R) \not\subseteq f_c(s^{i+j} R)$ .  $\square$

Note, there are still semiprime operations over the set of fractional ideals which are not star operations. For example, for any domain which is not Dedekind, the integral closure is a semiprime operation which is not prime and hence not a star operation.

### 3. PRIME OPERATIONS OVER ONE-DIMENSIONAL SEMIGROUP RINGS

In [Va], the author showed that the only prime operation over the set of ideals of a DVR (Proposition 3.4), a Dedekind domain (Proposition 3.7), and the cuspidal cubic,  $K[[t^2, t^3]]$  (Theorem 4.12) is the identity. We will show that if  $R = K[[t^2, t^{2n+1}]]$ , for  $n \geq 1$  then the only prime operation is the identity. However, we believe that if  $R = K[[t^S]]$ , for any one-dimensional semigroup  $S$  having ideals minimally generated by three or more natural numbers, then  $R$  has prime operations which are not the identity. In fact, we show that there is a non-identity prime operation on the set of ideals of  $K[[t^3, t^4, t^5]]$ .

We will determine a nice set of generators for all the ideals in  $R = K[[t^2, t^{2n+1}]]$  and the various relationships between the ideals. Similar to Proposition 4.1 in [Va], we will determine a nice form for the generator of a principal ideal. Then we will show that every ideal in  $R$  is minimally generated by two elements. First, we will look at the case  $R = K[[t^2, t^5]]$ .

**Proposition 3.1.** *Each nonzero nonunit principal ideal of  $R = K[[t^2, t^5]]$  can either be expressed in the form  $(t^2 + at^5)$ ,  $a \in K$  or  $(t^n + at^{n+1} + bt^{n+3})$ ,  $a, b \in K$ ,  $n \geq 4$ .*

PROOF. Suppose  $0 \neq f \in R$  and  $(f) \neq R$ . After multiplying by a nonzero element of  $K$ , either

- (1)  $f = t^2 + a_2 t^4 + a_3 t^5 + \dots$  or
- (2)  $f = t^n + a_1 t^{n+1} + a_2 t^{n+2} + \dots$  for  $n \geq 4$ .

Note that in case (1),  $n = 2$  and  $a_1$  must be 0. In both cases,  $f = t^2g$  where  $g \in K[[t^2, t^3]]$ . We saw in [Va] that if  $g \neq 0$  is not a unit, then  $(g) = (t^{n-2} + a_1t^{n-1})$  for  $n \geq 2$ . Thus  $g = (t^{n-2} + at^{n-1})u$ , where  $u$  is a unit in  $K[[t^2, t^3]]$ . We would like to show that  $(f) = (t^n + at^{n+1} + bt^{n+3})$  for some  $a, b \in K$ .

First, we will show that  $t^{n+2} + a_1t^{n+3} \in (f)$ . Note that

$$t^{n+2} + a_1t^{n+3} = t^2(t^n + a_1t^{n+1}) = (t^n + a_1t^{n+1})ut^2u^{-1} = ft^2u^{-1}.$$

Now since  $u^{-1}$  is a unit in  $K[[t^2, t^3]]$ , then  $t^2u^{-1} \in R$ . Similarly,  $t^{n+2} + a_1t^{n+3} \in (t^n + a_1t^{n+1} + (a_3 - a_1a_2)t^{n+3})$ .

Note if  $n \geq 4$ ,  $f = t^nh$  where  $h \in K[[t]]$  is a unit. We will see that  $t^r \in (f)$  for  $r \geq n + 4$  since  $t^r = t^{r-n}h^{-1}f$ . Similarly,  $t^r \in (t^n + a_1t^{n+1} + (a_3 - a_1a_2)t^{n+3})$ . Since  $f - (t^n + a_1t^{n+1} + (a_3 - a_1a_2)t^{n+3}) = a_2t^{n+2} + a_1a_2t^{n+3} + a_4t^{n+4} + a_5t^{n+5} + \dots$  is a member of  $(f) \cap (t^n + a_1t^{n+1} + (a_3 - a_1a_2)t^{n+3})$ , then

$$(f) = (t^n + a_1t^{n+1} + (a_3 - a_1a_2)t^{n+3}).$$

Hence, either  $(f) = (t^2 + at^5)$  for some  $a \in K$  or there are  $a, b \in K$  with  $(f) = (t^n + at^{n+1} + bt^{n+3})$  for  $n \geq 4$ .  $\square$

Using the fact that the principal ideals have the form  $(t^n + at^{n+1} + bt^{n+3})$  we can classify all the non-principal ideals of  $K[[t^2, t^5]]$ .

**Proposition 3.2.** *Each nonzero ideal of  $R = K[[t^2, t^5]]$  which is not principal is minimally generated by two elements of  $R$  and can either be expressed in the form  $(t^n, t^{n+1})$ ,  $n \geq 4$ ,  $(t^2, t^5)$ ,  $(t^n + at^{n+1}, t^{n+3})$  or  $(t^n, t^{n+1})$  for any  $a, b \in K$ ,  $n \geq 4$ .*

PROOF. The principal ideals are of the form  $(t^2 + at^5)$  or  $(t^n + at^{n+1} + bt^{n+3})$  by Proposition 3.1. Also  $t^{n+r} \in (t^n + at^{n+1} + bt^{n+3})$  for  $r \geq 4$ , if  $(t^n + at^{n+1} + bt^{n+3})$  is of minimal initial degree in  $I$ . Thus, we need only consider ideals  $I$  with additional generators of the form

- $t^n + ct^{n+1} + dt^{n+3}$ ,
- $t^{n+1} + ct^{n+2} + dt^{n+3}$ ,
- $t^{n+2} + ct^{n+3}$  and
- $t^{n+3}$ .

Suppose  $I$  has  $m - 1$  additional generators as above, then as

$$V = \{a_0t^n + a_1t^{n+1} + a_2t^{n+2} + a_3t^{n+3} \mid a_i \in K\}$$

is a  $K$ -vector subspace of  $K[[t^2, t^5]]$ , we can put the coefficients in rows of a  $m \times 4$  matrix  $A$  and put  $A$  into reduced row echelon form to help determine the minimal number of generators of  $I$ . The possible reduced row echelon forms for  $A$  are

$$\begin{aligned}
(1) \begin{pmatrix} 1 & a & 0 & b \\ 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 \end{pmatrix}, (2) \begin{pmatrix} 1 & 0 & B & C \\ 0 & 1 & D & E \\ 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 \end{pmatrix}, (3) \begin{pmatrix} 1 & B & 0 & C \\ 0 & 0 & 1 & D \\ 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 \end{pmatrix}, (4) \begin{pmatrix} 1 & B & C & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 \end{pmatrix}, \\
(5) \begin{pmatrix} 1 & 0 & 0 & B \\ 0 & 1 & 0 & C \\ 0 & 0 & 1 & D \\ 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 \end{pmatrix}, (6) \begin{pmatrix} 1 & 0 & B & 0 \\ 0 & 1 & C & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 \end{pmatrix}, (7) \begin{pmatrix} 1 & B & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 \end{pmatrix}, (8) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 \end{pmatrix}.
\end{aligned}$$

Now we use the properties of our ring  $R$  to simplify further. (1) represents a principal ideal so we can rule this case out.

(2) represents the ideal  $I = (t^n + Bt^{n+2} + Ct^{n+3}, t^{n+1} + Dt^{n+2} + Et^{n+3})$ . As

$$t^2(t^n + Bt^{n+2} + Ct^{n+3}) - (Bt^{n+4} + Ct^{n+5}) = t^{n+2} \in I,$$

then  $(t^n + Ct^{n+3}, t^{n+1} + Et^{n+3}) \subseteq I$ . Similarly,

$$t^2(t^{n+1} + Et^{n+3}) - Et^{n+5} = t^{n+3} \in I$$

implies that  $(t^n, t^{n+1}) \subseteq I$ . Clearly,  $t^{n+2}, t^{n+3} \in (t^n, t^{n+1})$  also implying that  $I = (t^n, t^{n+1})$ .

(3) represents the ideal  $I = (t^n + Bt^{n+1} + Ct^{n+3}, t^{n+2} + Dt^{n+3})$ . As

$$t^2(t^n + Bt^{n+1} + Ct^{n+3}) - (t^{n+2} + Dt^{n+3}) - Ct^{n+5} = (B - D)t^{n+3} \in I,$$

we now observe that if  $B - D \neq 0$ , then  $t^{n+3}$  and hence  $t^{n+2}$  and  $t^n + Bt^{n+1}$  are in  $I$  and clearly  $I \subseteq (t^n + Bt^{n+1}, t^{n+3})$  implying that  $I = (t^n + Bt^{n+1}, t^{n+3})$ . In the case that  $B - D = 0$ , then  $I$  is principal generated by  $t^n + Bt^{n+1} + Ct^{n+3}$ .

(4) represents the ideal  $I = (t^n + Bt^{n+1} + Ct^{n+2}, t^{n+3})$ . Since

$$t^2(t^n + Bt^{n+1} + Ct^{n+2}) - (Bt^{n+3} + Ct^{n+5}) = t^{n+2} \in I$$

and similarly  $t^{n+2} \in (t^n + Bt^{n+1} + Ct^{n+2}, t^{n+3})$  using similar reasoning to (2) above we see that  $I = (t^n + Bt^{n+1}, t^{n+3})$ .

(5) represents the ideal  $I = (t^n + Bt^{n+3}, t^{n+1} + Ct^{n+3}, t^{n+2} + Dt^{n+3})$ . Note that

$$t^{n+2} + Dt^{n+3} = t^2(t^n + Bt^{n+3}) + Dt^2(t^{n+1} + Ct^{n+3}) - (B + DC)t^{n+5}.$$

As we argued in (2) above, we see that  $I = (t^n, t^{n+1})$ .

(6) represents the ideal  $I = (t^n + Bt^{n+2}, t^{n+1} + Ct^{n+2}, t^{n+3})$ . As

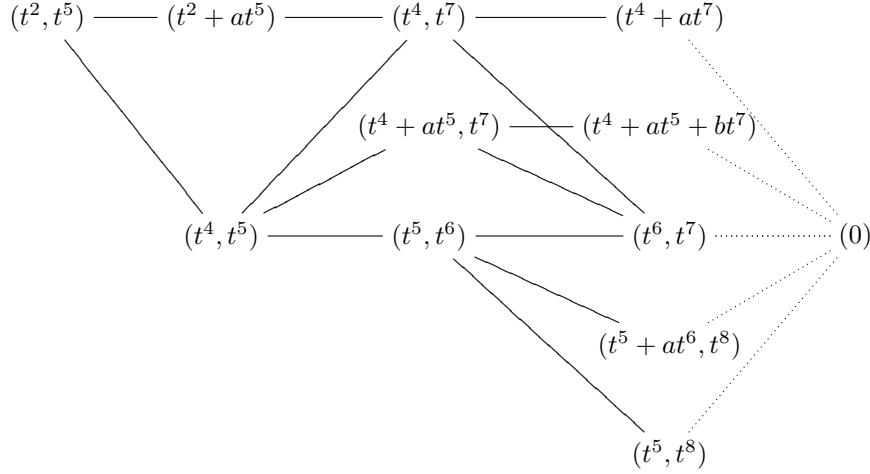
$$t^{n+2} = t^2(t^n + Bt^{n+2}) - Bt^{n+4} \in I$$

and  $t^{n+3} = t^2t^{n+1}$ ,  $I$  is minimally generated by  $(t^n, t^{n+1})$ .

(7) represents the ideal  $I = (t^n + Bt^{n+1}, t^{n+2}, t^{n+3})$ . We can conclude that  $I = (t^n + Bt^{n+1}, t^{n+3})$ , since  $t^{n+2} = t^2(t^n + Bt^{n+2}) - Bt^{n+3}$ .

(8) represents the ideal  $I = (t^n, t^{n+1}, t^{n+2}, t^{n+3})$ . Clearly,  $I$  is minimally generated by  $t^n$  and  $t^{n+1}$ .  $\square$

In fact a lattice of the proper ideals of  $K[[t^2, t^5]]$  is as follows:



where the lines indicate  $\supseteq$ . At each node where there is an  $a \in K$  present in the expression of the ideal, there are a cardinality of  $K$ 's worth of ideals at that node. If both  $a, b \in K$  are present, there are a cardinality of  $K^2$ 's worth of ideals at that node.

We can similarly determine the ideal structure of  $K[[t^2, t^{2r+1}]]$ . First we express the principal ideals similar to the way we expressed them in Proposition 3.1. In fact, the proof is very similar, building up on rings of the form  $K[[t^2, t^{2m+1}]]$  for  $m < r$ .

**Proposition 3.3.** *Each nonzero nonunit principal ideal of  $R = K[[t^2, t^{2r+1}]]$  can be expressed in one of the forms:  $(t^2 + a_1t^{2r+1}), (t^4 + a_1t^{2r+1} + a_3t^{2r+3}), \dots, (t^{2r-2} + a_1t^{2r+1} + \dots + a_{2r-3}t^{4r-3})$  or  $(t^n + a_1t^{n+1} + a_3t^{n+3} + \dots + a_{2r-1}t^{n+2r-1}), n \geq 2r$  and  $a_i \in K$ .*



PROOF. The proof is similar to Proposition 3.1. Suppose  $0 \neq f \in R$  and  $(f) \neq R$ . After multiplying by a nonzero element of  $K$ , either

- (1)  $f = t^{2k} + a_2 t^{2k+2} + \cdots + a_{2(r-k)} t^{2r} + a_{2(r-k)+1} t^{2r+1} + \cdots$  for  $1 \leq k < r$   
or  
(2)  $f = t^n + a_1 t^{n+1} + a_2 t^{n+2} + \cdots$  for  $n \geq 2r$ .

Note that in case (1),  $n = 2k$  for  $1 \leq k < r$  and  $a_1 = a_3 = a_{2(r-k)+1} = 0$ . In both cases,  $f = t^2 g$  where  $g \in K[[t^2, t^{2r-1}]]$ . We assume by induction that the principal ideals of  $K[[t^2, t^{2r-1}]]$  are of the form  $(t^2 + a_1 t^{2r-1})$ ,  $(t^4 + a_1 t^{2r-1} + a_3 t^{2r+1})$ ,  $\dots$ ,  $(t^{2r-4} + a_1 t^{2r-1} + \cdots + a_{2r-5} t^{4r-7})$  or  $(t^n + a_1 t^{n+1} + a_2 t^{n+3} + \cdots + a_{2r-3} t^{n+2r-3})$ ,  $n \geq 2(r-1)$  and  $a_i \in K$ . As in Proposition 3.1 if  $g \neq 0$  is not a unit, then  $(g) = (t^{n-2} + a_1 t^{n-1} + \cdots + a_{2r-5} t^{n+2r-3})$  for  $n \geq 2$ . Thus  $g = (t^{n-2} + a_1 t^{n-1} + a_3 t^{n+1} + \cdots + a_{2r-5} t^{n+2r-3})u$ , where  $u$  is a unit in  $K[[t^2, t^{2r-1}]]$ . We show that  $(f) = (t^n + a_1 t^{n+1} + a_3 t^{n+3} + \cdots + a_{2r-1} t^{n+2r-1})$  for  $a_i \in K$ .

First, we will show that  $t^{n+2} + a_1 t^{n+3} + a_{2r-3} t^{n+2r-1} \in (f)$ . Note that

$$\begin{aligned} f t^2 u^{-1} &= t^2 (t^n + a_1 t^{n+1} + a_{2r-3} t^{n+2r-3}) u t^2 u^{-1} \\ &= t^{n+2} + a_1 t^{n+3} + a_{2r-3} t^{n+2r-1}. \end{aligned}$$

Now since  $u^{-1}$  is a unit in  $K[[t^2, t^{2r-1}]]$ , then  $t^2 u^{-1} \in R$ . Similarly,

$$t^{n+2} + a_1 t^{n+3} + a_{2r-3} t^{n+2r-1} \in (t^n + a_1 t^{n+1} + b_3 t^{n+3} + \cdots + b_{2r-1} t^{n+2r-1})$$

where  $b_3 = a_3 - a_2 a_1$ ,  $b_5 = a_5 - a_4 a_1 - a_2 a_3 + a_1 (a_2)^2$  and generally  $b_{2m-1}$  is

$$a_{2m-1} - \sum_{i=1}^{m-1} a_{2i-1} a_{2(m-i)} + \sum_{i=1}^{m-2} \sum_{j=1}^{m-i} a_{2i-1} a_{2j} a_{2(m-i-j)} - \cdots + (-1)^{m-1} a_1 a_2^{m-1}.$$

Note if  $n \geq 2r$ ,  $f = t^n h$  where  $h \in K[[t]]$  is a unit. We will see that  $t^m \in (f)$  for  $m \geq n + 2r$  since  $t^m = t^{m-n} h^{-1} f$ . Similarly,  $t^m$  is belongs to the ideal  $(t^n + a_1 t^{n+1} + (a_3 - a_1 a_2) t^{n+3})$ . Since  $f - (t^n + a_1 t^{n+1} + b_3 t^{n+3} + \cdots + b_{2r-1} t^{n+2r-1})$  is a member of  $(f) \cap (t^n + a_1 t^{n+1} + b_3 t^{n+3} + \cdots + b_{2r-1} t^{n+2r-1})$ , then the ideals  $(f)$  and  $(t^n + a_1 t^{n+1} + b_3 t^{n+3} + \cdots + b_{2r-1} t^{n+2r-1})$  are equal. Hence, the principal ideals are those stated in the Proposition.  $\square$

Using the structure of the principal ideals which we determined in Proposition 3.3, we can now determine the structure of all non-principal ideals.

**Proposition 3.4.** *Each nonzero ideal of  $R = K[[t^2, t^{2r+1}]]$  which is not principal is minimally generated by two elements of  $R$  and can be expressed in the one of the following forms:  $(t^2, t^{2r+1})$ ,  $(t^4, t^{2r+1})$ ,  $(t^4 + a_1 t^{2r+1}, t^{2r+3})$ ,  $\dots$ ,  $(t^{2r-2} + a_1 t^{2r+1} + \cdots + a_{r-1} t^{4r-5}, a_r t^{4r-3})$ , or  $(t^n, t^{n+1})$ ,  $((t^n + a_1 t^{n+1}, t^{n+3})$ ,*

$(t^n + a_1 t^{n+1} + a_3 t^{n+3} + \dots + a_{2r-3} t^{n+2r-3}, t^{n+2r-1})$ ,  $n \geq 2r$  and  $a_{2m+1} \in K$  for  $0 \leq m \leq r$ .

PROOF. The proof is similar to the proof of Proposition 3.2, albeit slightly more technical. We will only give a sketch of it here. As in the proof of Proposition 3.2, we can determine the possible generators for an ideal  $I$  containing the element  $f = t^n + a_1 t^{n+1} + \dots + a_r t^{n+2r-1}$  where  $f$  was an element of  $I$  with a minimal initial term. Then we will put the coefficients in an  $m \times 2r$  matrix  $A$  and determine all reduced row echelon expressions for  $A$ . After we do this, we use the properties of the ring to see that all the ideals will have the structure as in the statement of the Proposition.  $\square$

Now we will see that the only prime operation on the ring  $K[[t^2, t^{2r+1}]]$  is the identity.

**Proposition 3.5.** *The only prime operation on the set of ideals of  $R = K[[t^2, t^{2r+1}]]$  is the identity.*

PROOF. Since every prime operation satisfies

$$(1) \quad f_c((g)I) = (g)f_c(I)$$

for all principal ideals  $(g) \in R$ , if we substitute  $R$  in for  $I$  in (1), we obtain  $f_c((g)) = (g)$ . Thus all principal ideals must be  $f_c$ -closed. Note that each principal ideal is sandwiched between many 2-generated ideals. Suppose  $f_c(I) \neq I$  for some two generated ideal

$$I = (t^n + a_1 t^{n+1} + \dots + a_{2m-3} t^{n+2m-3}, t^{n+2m-1})$$

for  $2 \leq m \leq r$ . Then  $f_c((g)I) = (g)f_c(I) \neq (g)I$  for any principal ideal  $(g)$ . Since the conductor of  $R$  is  $\mathfrak{c} = (t^{2r}, \dots, t^{4r-1})$ . Then for any  $(g) \in \mathfrak{c}$ ,  $f_c((g)I) \neq (g)I$ . Hence, all the ideals of the form  $J_{k,S} = (t^k + b_1 t^{k+1} + \dots + b_{2m-3} t^{k+2m-3}, t^{k+2m-1})$  for  $k \geq n + 2r$  and  $S$  the  $m - 1$ -tuple  $(b_1, \dots, b_{m-1})$  satisfy  $f_c(J_{k,S}) \neq J_{k,S}$ . Note  $f_c(J_{k,S}) \supseteq (t^{k-2r+2m-2} + b_1 t^{k-2r+2m-1} + \dots + b_{2m-3} t^{k-2r+4m-5} + b t^{k-2r+4m-3})$  some  $b \in K$  or

$$f_c(J_{k,S}) \supseteq (t^k + b_1 t^{k+1} + \dots + b_{k+2m-5} t^{k+2m-5}, t^{k+2m-3}).$$

If  $f_c(J_{k,S}) = (t^k + b_1 t^{k+1} + \dots + b_{k+2m-5} t^{k+2m-5}, t^{k+2m-3})$ , then since

$$(t^{k-2r+2m-2} + b_1 t^{k-2r+2m-1} + \dots + b_{2m-3} t^{k-2r+4m-5} + b t^{k-2r+4m-3})$$

is incomparable to

$$(t^k + b_1 t^{k+1} + \dots + b_{k+2m-5} t^{k+2m-5}, t^{k+2m-3})$$

and

$$f_c((t^{k-2r+2m-2} + b_1 t^{k-2r+2m-1} + \dots + b_{2m-3} t^{k-2r+4m-5} + b t^{k-2r+4m-3}))$$

contains  $(t^k + b_1 t^{k+1} + \dots + b_{k+2m-5} t^{k+2m-5}, t^{k+2m-3})$ , this forces

$$f_c((t^{k-2r+2m-2} + b_1 t^{k-2r+2m-1} + \dots + b_{2m-3} t^{k-2r+4m-5} + b t^{k-2r+4m-3}))$$

to be not equal to

$$(t^{k-2r+2m-2} + b_1 t^{k-2r+2m-1} + \dots + b_{2m-3} t^{k-2r+4m-5} + b t^{k-2r+4m-3}),$$

contradicting that  $f_c(g) = (g)$  for all principal  $(g)$ .

If  $f_c(J_{k,S}) = (t^{k-2r+2m-2} + b_1 t^{k-2r+2m-1} + \dots + b_{2m-3} t^{k-2r+4m-5} + b t^{k-2r+4m-3})$  then

$$f_c((t^k + b_1 t^{k+1} + \dots + b_{k+2m-5} t^{k+2m-5}, t^{k+2m-3}))$$

contains

$$(t^{k-2r+2m-2} + b_1 t^{k-2r+2m-1} + \dots + b_{2m-3} t^{k-2r+4m-5} + b t^{k-2r+4m-3})$$

which will also force

$$f_c((t^{k+2r-2m+4} + b_1 t^{k+2r-2m+5} + \dots + b_{k+2m-5} t^{k+2r-1}, t^{k+2r+1}))$$

to contain

$$(t^{k+2} + b_1 t^{k+3} + \dots + b_{2m-3} t^{k+2m-1} + b t^{k+2m+1}).$$

Now since  $(t^k + b_1 t^{k+1} + \dots + b_{k+2m-5} t^{k+2m-5}, t^{k+2m-3})$  is incomparable to

$$(t^{k-2r+2m-2} + b_1 t^{k-2r+2m-1} + \dots + b_{2m-3} t^{k-2r+4m-5} + b t^{k-2r+4m-3})$$

and the fact that  $(t^k + b_1 t^{k+1} + \dots + b_{k+2m-5} t^{k+2m-5}, t^{k+2m-3})$  contains

$$(t^k + b_1 t^{k+1} + \dots + b_{k+2m-5} t^{k+2m-5} + c t^{k+2m-3})$$

which contains

$$(t^{k+2r-2m+4} + b_1 t^{k+2r-2m+5} + \dots + b_{k+2m-5} t^{k+2r-1}, t^{k+2r+1}),$$

then we see that  $f_c((t^k + b_1 t^{k+1} + \dots + b_{k+2m-5} t^{k+2m-5} + c t^{k+2m-3}))$  is not equal to

$$(t^k + b_1 t^{k+1} + \dots + b_{k+2m-5} t^{k+2m-5} + c t^{k+2m-3})$$

some  $c \neq b_{2m-3}$ , again contradicting that  $f_c(g) = (g)$  for all principal  $(g)$ .  $\square$

Although, there are no non-identity prime operations over semigroup rings which have ideals minimally generated by at most two elements. We can show that there is a prime operation on  $k[[t^3, t^4, t^5]]$  which is not the identity. This answers a question posed on the last page of [Va] in the negative. We suspect that in semigroup rings which have ideals minimally generated by 3 or more elements will also have prime operations which are not the identity. First, we generators for all the ideals of  $k[[t^3, t^4, t^5]]$ .

**Proposition 3.6.** *Let  $R = k[[t^3, t^4, t^5]]$ . Then the nonzero, nonunit ideals of  $R$  can be expressed in the forms  $(t^n + at^{n+1} + bt^{n+2})$ ,  $(t^n + at^{n+1}, t^{n+2})$ ,  $(t^n + at^{n+2}, t^{n+1} + bt^{n+2})$  or  $(t^n, t^{n+1}, t^{n+2})$  for  $n \geq 3$  and  $a, b \in k$ .*

PROOF. In [FV], Fouli and the current author showed in Proposition 5.2 that all nonunit principal ideals are of the form  $(t^n + at^{n+1} + bt^{n+2})$ , for  $n \geq 3$  and  $a, b \in k$ .

Now we consider the form of all non-principal ideals  $I$ . Since  $(t^{n+3}, t^{n+4}, t^{n+5}) \subseteq (t^n + at^{n+1} + bt^{n+2})$ , if  $(t^n + at^{n+1} + bt^{n+2}) \in I$  is the form with the initial term of smallest degree, the other generators can only be of the form  $(t^n + ct^{n+1} + dt^{n+2}, t^{n+1} + et^{n+2})$  and  $t^{n+2}$ .

Observe that  $V = \{a_0t^n + a_1t^{n+1} + a_2t^{n+2} \mid a_i \in k, n \geq 3\}$  is a subspace of the  $k$ -vector space  $k[[t^3, t^4, t^5]]$ . To determine a minimal set of generators for  $I$ , we can set the coefficients of the  $m$  generators of  $I$  in rows of an  $m \times 3$  matrix  $A$ . The possible reduced row echelon forms for  $A$  are:

$$\begin{pmatrix} 1 & a & b \\ 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & B \\ 0 & 1 & C \\ 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & B & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \end{pmatrix}$$

for  $B$  and  $C$  any elements of  $k$ . These row echelon forms correspond to the ideals listed in the statement of the proposition.  $\square$

**Proposition 3.7.** *There is a prime operation  $f_c$  on the set of ideals of  $R = k[[t^3, t^4, t^5]]$  which is not the identity.*

PROOF. The principal ideal  $(f) = (t^n + at^{n+1} + bt^{n+2})$  is contained in all the two-generated ideals containing

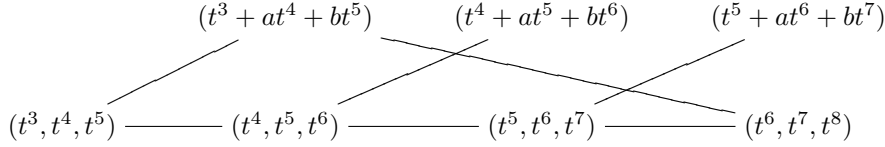
$$(t^n, t^{n+1} + \frac{b}{a}t^{n+2}), (t^n + at^{n+1}, t^{n+2}), (t^n + ct^{n+2}, t^{n+1} + \frac{b-c}{a}t^{n+2}),$$

for all  $c \in k$  as long as  $0 \neq a \in k$ .  $(f)$  is also contained in all the three generated ideals containing  $(t^n, t^{n+1}, t^{n+2})$ . The ideal  $(t^n + at^{n+1} + bt^{n+2})$  contains all ideals contained in  $(t^{n+3}, t^{n+4}, t^{n+5})$ .

The following ideals are incomparable to  $(t^n + at^{n+1} + bt^{n+2})$ :

- 3-generated ideals:  $(t^{n+1}, t^{n+2}, t^{n+3})$ ,  $(t^{n+2}, t^{n+3}, t^{n+4})$ ,
- 2-generated ideals with initial degree greater than  $n$ :  
 $(t^{n+1}, t^{n+2} + dt^{n+3})$ ,  $(t^{n+1} + dt^{n+2}, t^{n+3})$ ,  $(t^{n+1} + dt^{n+3}, t^{n+2} + et^{n+3})$ ,  
 $(t^{n+2}, t^{n+3} + dt^{n+4})$ ,  $(t^{n+2} + dt^{n+3}, t^{n+4})$  and  $(t^{n+2} + dt^{n+4}, t^{n+3} + et^{n+4})$   
for any  $d, e \in k$ ,
- 2-generated ideals with initial degree less than or equal to  $n$ :  
 $(t^n, t^{n+1} + dt^{n+2})$  or  $(t^{n-2} + dt^{n-1}, t^n)$  with  $d \neq \frac{b}{a}$ ,  $(t^n + dt^{n+1}, t^{n+2})$  with  
 $d \neq a$ ,  $(t^n + dt^{n+2}, t^{n+1} + et^{n+2})$  with  $ae \neq b - d$ ,  $(t^{n-1} + dt^n, t^{n+1})$  for  
any  $d \in k$ ,  $(t^{n-1} + dt^{n+1}, t^n + et^{n+2})$  for any  $d \in k$  and  $a \neq e \in k$  and  
 $(t^{n-2} + dt^n, t^{n-1} + et^n)$  for any  $d, e \in k$ .
- principal ideals:  $(t^n + dt^{n+1} + et^{n+2})$  for  $(a, b) \neq (d, e) \in k^2$ ,  
 $(t^{n+i} + dt^{n+i+1} + et^{n+i+2})$  for any  $d, e \in k$  and  $i \in \{-2, -1, 1, 2\}$ .

Depicting a lattice of all ideals and their relationships with each other would be quite complicated, but we can easily draw part of the lattice of three generated ideals and principal ideals:



where each of the lines denote  $\supseteq$  and at each principal node, there are the cardinality of  $k^2$  ideals.

We will use this lattice to define a prime operation  $f_c$  on the ideals of  $R$  as follows:  $f_c(I) = (t^n, t^{n+1}, t^{n+2})$  if  $I = (t^n, t^{n+1} + ct^{n+2})$ ,  $I = (t^n + dt^{n+1}, t^{n+2})$ ,  $I = (t^n + dt^{n+1}, t^{n+1} + et^{n+2})$  or  $I = (t^n, t^{n+1}, t^{n+2})$ . However,

$$f_c((t^n + at^{n+1} + bt^{n+2})) = (t^n + at^{n+1} + bt^{n+2}).$$

Now we verify that  $f_c$  is a prime operation. Clearly,  $I \subseteq f_c(I)$ . Also since any ideal  $I$  containing  $(t^n + at^{n+1} + bt^{n+2})$  has the property that  $f_c(I) \supseteq (t^n, t^{n+1}, t^{n+2})$  and any ideal contained in  $(t^n + at^{n+1} + bt^{n+2}) \supseteq (t^{n+3}, t^{n+4}, t^{n+5})$  has the property that  $f_c(I) \subseteq (t^{n+3}, t^{n+4}, t^{n+5}) \subseteq (t^n + at^{n+1} + bt^{n+2})$  then if  $I \subseteq J$ , then  $f_c(I) \subseteq f_c(J)$ . Clearly,  $f_c(f_c(I)) = f_c(I)$  because if  $I$  is principal

then  $f_c(I) = I$  and if  $I$  is not principal then  $f_c(I) = (t^n, t^{n+1}, t^{n+2})$  for some  $n$  and  $(t^n, t^{n+1}, t^{n+2})$  is  $f_c$ -closed.

Note that the product of two principal ideals is principal, thus  $f_c((f)(g)) = f_c((f))f_c((g))$  for principal ideals  $(f)$  and  $(g)$ . The product of any ideal with a non-principal ideal will be non-principal. If  $I = (t^n + at^{n+1} + bt^{n+2}) + I'$  and  $J = (t^m + ct^{m+1} + dt^{m+2}) + J'$  with  $I'$  incomparable to  $(t^n + at^{n+1} + bt^{n+2})$  and  $J'$  incomparable to  $(t^m + ct^{m+1} + dt^{m+2})$  or the zero ideal, then  $IJ$  is equal to  $(t^{m+n} + (a+c)t^{m+n+1} + (ac+b+d)t^{m+n+2}) + K'$  where

$$K' = (t^n + at^{n+1} + bt^{n+2})J' + (t^m + ct^{m+1} + dt^{m+2})I' + I'J'$$

which is minimally generated by either 2 or 3 elements. In any case  $f_c(IJ) = (t^{m+n}, t^{m+n+1}, t^{m+n+2}) = f_c(I)f_c(J)$ . Hence,  $f_c$  is a prime operation which is not the identity. □

It seems very likely, that we should be able to exhibit non-identity prime operations in one-dimensional semigroup rings with ideals minimally generated by three or more elements using an argument similar to the one above.

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