10.21 (b) $n^2 \le 2^n$

Let $P(n)$ be the statement $n^2 \le 2^n$. $n \in \mathbb{N}$

Note : $P(1)$ asserts $1^2 \le 2^1$
$P(2)$ asserts $2^2 \le 2^2$ } which are true
$P(3)$ asserts $3^2 \le 2^3$ – which is not true
$P(4)$ asserts $4^2 \le 2^4$ – which is true

Now, assume $P(n)$ is true for $n \ge 4$

$P(n+1)$ asserts $(n+1)^2 \le 2^{n+1}$

$$n^2 + 2n + 1 \le 2 \cdot 2^n$$

Now note that $2n^2 \le 2 \cdot 2^n$ ... (by induction hypothesis)

$$n^2 + n^2 \le 2 \cdot 2^n$$

Now note that, for $n \ge 4$,

$$n^2 \ge 4n$$

$$n^2 \ge 2n + 2n \ge 2n+1$$

Combining this information:

$$n^2 + 2n + 1 \le n^2 + n^2 \le 2 \cdot 2^n$$

$$(n+1)^2 \le 2^{n+1}$$

Thus, $P(n+1)$ is true whenever $P(n)$ is true

& by induction we conclude that $P(n)$ is true for all $n \ge 4$ and $n = 1, n$

11.4 (a) if $x \ge 0$ and $x \le \varepsilon$, $\forall\, \varepsilon > 0$, then $x = 0$

By theorem 11.7, if $x \le 0 + \varepsilon = \varepsilon$ for every $\varepsilon > 0$ then $x \le 0$. But $x \ge 0$. Now since $\le$ is a partial ordering on $\mathbb{R}$, antisymmetry implies $x = 0$.