#6

11.6 (a) $||x|-|y|| \le |x-y|$

First note that $|x-y| = |y-x|$ since $|x-y| = \begin{cases} x-y & x>y \\ y-x & x<y \end{cases} = |y-x|$

By the Δ-inequality, $|x-y|+|y| \ge |x-y+y| = |x|$ and $|y-x|+|x| \ge |y|$

Thus, $|x-y| \ge |x|-|y|$ and $|x-y| = |y-x| \ge |y|-|x|$ by O3.

Now $||x|-|y|| = \begin{cases} |x|-|y| & \text{if } |x|>|y| \\ |y|-|x| & \text{if } |y|>|x| \end{cases}$. Since $|x-y| \ge |x|-|y|$ and $|x-y| \ge$

For all $x, y \in \mathbb{R}$, then $|x-y| \ge ||x|-|y||$.

(b) If $|x-y| < c$, then $|x| < |y| + c$

pf: NOTE:

$$|x| - |y| = |x-y+y| - |y| \le |x-y| + |y| - |y| = |x-y|$$

... (by the TRIANGLE INEQUALITY)

Now, Assume $|x| \ge |y| + c$

$$|x| - |y| \ge c$$

$$|x-y| \ge |x| - |y| \ge c \quad \text{... (by the above note)}$$

Thus, $|x-y| \ge c$ and we have shown that if $|x-y| < c$, then $|x| < |y| + c$ by the contrapositive. ✓

(c) If $|x-y| < \varepsilon$ for all $\varepsilon > 0$, then $x = y$.

pf: Assume $x \ne y$. ✓

Then $|x-y| > 0$

Now, Let $\varepsilon = \frac{|x-y|}{2} > 0$

Then, $0 < \varepsilon \le |x-y|$

i.e. it has been shown that if $x \ne y$, then $\exists$ some ε such that $|x-y| \ge \varepsilon$

And, thus we can conclude if $|x-y| < \varepsilon$ for all $\varepsilon > 0$ then $x = y$, by the contrapositive.

note: ε is not unique

consider, $\frac{|x-y|}{3}$, $\frac{|x-y|}{4}$, etc...