$\mathfrak{m}$-FULL AND BASICALLY FULL IDEALS IN RINGS OF CHARACTERISTIC p

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Abstract. We generalize the notion of $\mathfrak{m}$- full and basically full ideals in the setting of tight closure and demonstrate some $\mathfrak{m}$-full and basically full ideals in non-regular rings.

1. Introduction

For simplicity, let $(R, \mathfrak{m})$ be Noetherian local ring of characteristic p. However, everything that we discuss in characteristic p can be generalized to an equicharacteristic Noetherian local ring.

After hearing some lectures by Rees in Japan in the late 80's, Junzo Watanabe wrote a paper on $\mathfrak{m}$-full ideals. In a ring with infinite residue field, he defined an ideal I to be $\mathfrak{m}$-*full* if $(\mathfrak{m}I : x) = I$ for some $x \in \mathfrak{m} \setminus \mathfrak{m}^2$. He showed that an ideal I which is $\mathfrak{m}$-full satisfies *the Rees Property*: $\mu(I) \geq \mu(J)$ for all $J \supseteq I$. In a normal domain, he proved that any integrally closed ideal is $\mathfrak{m}$-full [Wa, Theorem 5].

In regular rings, several authors have given criteria which help to determine if an ideal is $\mathfrak{m}$-full. In particular, Watanabe has shown the following for 2-dimensional regular local rings:

Theorem 1.1. [Wa, Theorem 4] *Let $(R, \mathfrak{m})$ be a 2-dimensional regular local ring, I an $\mathfrak{m}$-primary ideal with $I \subseteq \mathfrak{m}^n$ and $I \nsubseteq \mathfrak{m}^{n+1}$, the following are equivalent:*
(a) I is $\mathfrak{m}$-full.
(b) $\mu(I) = n + 1$
(c) I satisfies the Rees property.
(d) $(I : \mathfrak{m}) = (I : x)$ some $x \in \mathfrak{m}$.

In both [HuSw] and [HNLR], ideals, I, satisfying item (d) in the above theorem have been called *full*. Another result pertaining to $\mathfrak{m}$-full parameter ideals in regular local rings of any dimension can be summed up in the following Theorem:

Theorem 1.2. [Go, Proposition 2.3], [HNLR, Theorem 4.1] *Let $(R, \mathfrak{m})$ be a regular local ring, I be a parameter ideal, then the following are equivalent:*
(a) $I^n = \overline{I^n}$ for all $n \geq 1$ (I is normal.)
(b) I is integrally closed.
(c) I is $\mathfrak{m}$-full.
(d) I is full.

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(e) $\lambda((I+\mathfrak{m}^2)/\mathfrak{m}^2) \geq d-1$

There is not as much known about $\mathfrak{m}$-primary, $\mathfrak{m}$-full ideals in non-regular rings. However, the following are interesting results of Goto and Hayasaka [GH] and Ciuperca [Ciu]:

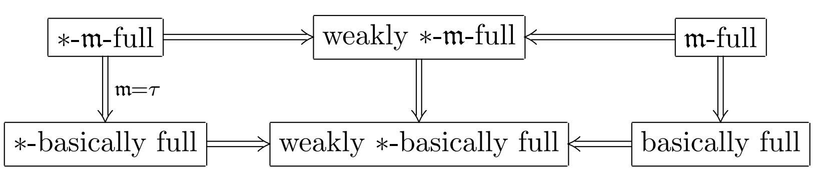
Proposition 1.3. [GH, Proposition 2.4] *Let $(R,\mathfrak{m})$ be a Noetherian local ring and I an $\mathfrak{m}$-primary ideal with $n=\mu(I)$, then the following are equivalent:*
(a) I is $\mathfrak{m}$-full and R/I Gorenstein.
(b) $\mu(\mathfrak{m})=n$ and there exists a minimal basis $a_1,\dots a_n$ of $\mathfrak{m}$ such that $I=(a_1,\dots,a_{n-1},a_n^s)$ for some $s=min\{r \mid \mathfrak{m}^r \subseteq I\}$.

Proposition 1.4. [Ciu, Proposition 4.1] *Let $(R,\mathfrak{m})$ be a d-dimensional Noetherian local ring with* $\operatorname{edim} R = d+1$ *Suppose I is an $\mathfrak{m}$-primary ideal generated by $d+1$ elements. I is $\mathfrak{m}$-full if and only if there exist generators $x,y,a_1,\dots,a_{d-1}$ of $\mathfrak{m}$ such that*
(a) $I+xR=\mathfrak{m}$ or
(b) $\mathfrak{m}^2=(x,a_1,\dots,a_{d-1})\mathfrak{m}$ and $I+xR=(x,a_1,\dots,a_{d-1})$.

However, neither of the above propositions give us a feeling for what $\mathfrak{m}$-full ideals look like which sit much deeper inside of the maximal ideal.

In 2002, Heinzer, Ratliff and Rush [HRR] defined the related concept: basically full ideals. An ideal is *basically full* if no minimal set of generators of I can be extended to a minimal set of generators for J, an ideal containing I. It was also shown in [HRR] that a basically full ideal is $\mathfrak{m}$-primary and satisfies $(\mathfrak{m}I:\mathfrak{m})=I$. Recall, that monomials in a regular local ring are partially ordered as follows $x_1^{a_1}\cdots x_d^{a_d} \leq x_1^{b_1}\cdots x_d^{b_d}$ if and only if $a_i \leq b_i$ for all $1 \leq i \leq d$. A set of monomials form an antichain if the generators are pairwise incomparable. In [HRR, Proposition 8.5], Heinzer, Ratliff and Rush give a nice criterion for determining if a monomial ideal in a regular local ring is basically full: a monomial ideal I is basically full if the minimal set of generators for I is a maximal antichain. Note that for $\mathfrak{m}$-primary ideals I, $(\mathfrak{m}I:\mathfrak{m}) \subseteq (\mathfrak{m}I:x)$ for all $x \in \mathfrak{m}\setminus\mathfrak{m}^2$; hence, all $\mathfrak{m}$-full ideals are basically full.

In this paper, we will use tight closure, in particular colon capturing, to illustrate a way to find $\mathfrak{m}$-full ideals and basically full ideals in non-regular rings. We also define tight closure notions of $\mathfrak{m}$-full and basically full ideals. In particular we can show:



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2. Tight Closure and Test Ideals

Tight closure, introduced by Hochster and Huneke, is a closure operation for rings containing a field. Let R be a ring of characteristic p. We say an element $x \in R$ is in the *tight* *closure* I^* of I if there exists a $c \in R \backslash \bigcup_{P \in \mathrm{Min}(R)} P$ with $cx^q \in I^{[q]}$ for all large $q = p^e$ where $I^{[q]}$ is the ideal generated by all the qth powers of elements in I. If $I = I^*$, we say that I is tightly closed. Tight closure has given easy proofs to some very hard problems in commutative algebra such as the Briançon-Skoda Theorem and many others. When Hochster and Huneke first defined tight closure, they noted that the tight closure I^* is contained in the integral closure. It sits much closer to I in general than the integral closure, hence, a tighter fit. Although, there are not many rings for which all ideals are integrally closed, there are many rings for which all ideals are tightly closed. We call a local ring $(R, \mathfrak{m})$ *weakly F-regular* if all ideals are tightly closed. Of course, all regular rings are weakly F-regular, but these are not the only rings which are weakly F-regular. For example, $R = k[[x, y, z]]/(x^2 - y^3 - z^5)$ is weakly F-regular when the characteristic of k is greater than 5.

Recall that if $x_1, \ldots, x_n$ is a regular sequence, then

$$(x_1, \ldots, \hat{x_i}, \ldots, x_n) : x_i = (x_1, \ldots, \hat{x_i}, \ldots, x_n).$$

In a Noetherian ring of characteristic $p > 0$ which is a homomorphic image of a Cohen Macaulay ring, we say that parameters $x_1, \ldots, x_n$ satisfy colon capturing if $(x_1, \ldots, x_{n-1}) : x_n \subseteq (x_1, \ldots, x_{n-1})^*$. Since complete local domains are always the homomorphic image of a Cohen Macaulay ring, we would like to mention an stronger version of colon capturing in this instance:

Theorem 2.1. [Hu1, Theorem 9.2] *Let $(R, \mathfrak{m})$ be a d-dimensional complete local domain of characteristic p with coefficient field k. Let $x_1, \ldots, x_d$ be a system of parameters for R and let I and J be ideals of the subring $A = k[[x_1, \ldots, x_d]]$. Then*
(a) $(IR)^* :_R JR \subseteq ((I : J)R)^*$ *and*
(b) $(IR)^* \cap (JR)^* \subseteq ((I \cap J)R)^*$.

As we saw from the definition of tight closure, it is necessary to have an element c which is not contained in any minimal prime, to find the elements in the tight closure of an ideal. How do we know if any given c will multiply a qth power of an element into $I^{[q]}$? We say an element c is a *test element* if $cI^* \subseteq I$ for all ideals $I \subseteq R$. Having test elements, enables us to compute the tight closure of an ideal. The test ideal, $\tau = \bigcap_{I \subseteq R} (I : I^*)$, is the ideal generated by all the test elements.

Huneke introduced the notion of strong test ideals in [Hu2]. An ideal J is a *strong test ideal*, if $JI^* = JI$ for all ideals I in R. Vraciu has shown [Vr] that the test ideal is a strong test ideal in a complete reduced ring. Hara and Smith have shown [HS] for a local ring $(R, \mathfrak{m})$, if $\mathfrak{m}$ is the test ideal $\mathfrak{m}$ is a strong test ideal.

The test ideal can be used effectively to compute the tight closure of a parameter ideal in a Gorenstein local ring.

Proposition 2.2. [Sm] *Let $(R, \mathfrak{m})$ be a Gorenstein local ring of dimension d with an $\mathfrak{m}$-primary test ideal τ and $x_1, x_2, \ldots, x_d$ be a system of parameters. Then*

$$(x_1, x_2, \ldots, x_d) : \tau = (x_1, x_2, \ldots, x_d)^*.$$

Also, when $(R, \mathfrak{m})$ is a Gorenstein local ring with $\mathfrak{m}$-primary test ideal, one can use a system of parameters to compute the test ideal.

Proposition 2.3. [Hu1, Exercise 2.14] *Let $(R, \mathfrak{m})$ be a Gorenstein ring of dimension d with $\mathfrak{m}$-primary test ideal τ and $x_1, x_2, \dots, x_d$ be a system of parameters which are also test elements. Then*

$$(x_1, x_2, \dots, x_d) : (x_1, x_2, \dots, x_d)^* = \tau.$$

The following proposition gives us a nice criterion for computing the tight closure of non-parameter ideals using the test ideal.

Proposition 2.4. *Let $(R, \mathfrak{m})$ be a complete Gorenstein local domain of dimension d with $\mathfrak{m}$ primary test ideal τ. Suppose $x_1, x_2, \dots, x_d$ is a system of parameters in R and I is an ideal of R which is the intersection of parameter ideals whose generators are in $k[[x_1, x_2, \dots, x_d]]$, then $(I : \tau) = I^*$.*

Proof: Let

$$I = \bigcap_{j \geq 1} (y_{j1}, y_{j2}, \dots, y_{jd})$$

where $y_{j1}, y_{j2}, \dots, y_{jd}$ are parameters. Note that

$$(y_{j1}, y_{j2}, \dots, y_{jd}) : \tau = (y_{j1}, y_{j2}, \dots, y_{jd})^*.$$

As

$$(I : \tau) = \bigcap_{j \geq 1} (y_{j1}, y_{j2}, \dots, y_{jd}) : \tau = \bigcap_{j \geq 1} (y_{j1}, y_{j2}, \dots, y_{jd})^*,$$

we see

$$I^* \subseteq \bigcap_{j \geq 1} (y_{j1}, y_{j2}, \dots, y_{jd})^*.$$

Now as $(y_{j1}, \dots, y_{jd}) \subseteq (x_1, \dots, x_d)$, then by Theorem 2.1,

$$\bigcap_{j \geq 1} (y_{j1}, y_{j2}, \dots, y_{jd})^* \subseteq I^*.$$

□

3. $\mathfrak{m}$-full and basically full ideals in complete domains

As mentioned in the introduction, for non regular rings, there is no criterion for finding $\mathfrak{m}$-full and basically full ideals deep inside the maximal ideal. One effective way for obtaining such ideals, are using the known criteria for finding $\mathfrak{m}$-full and basically full ideals deep inside the maximal ideal of a regular local ring and use colon capturing.

Theorem 3.1. *Let R be a complete local domain with coefficient field k and $x_1, \dots, x_d$ is a system of parameters. Let $\mathfrak{n}$ be the maximal ideal of $A = k[[x_1, \dots, x_d]]$. Suppose $I \subseteq k[[x_1, \dots, x_d]]$ and $(\mathfrak{n}R)^* = \mathfrak{m}$.*

(1) If I is $\mathfrak{n}$-full, then $(IR)^*$ is $\mathfrak{m}$-full.
(2) If I is basically full in A, then $(IR)^*$ is basically full in R.

Proof: For (1), note that for some $x \in \mathfrak{n} \setminus \mathfrak{n}^2$, $(\mathfrak{n}I :_A x) = I$ and

$$\begin{aligned}(IR)^* &\subseteq (\mathfrak{m}(IR)^* :_R x)\\ &= ((\mathfrak{n}R)^*(IR)^* :_R x)\\ &\subseteq (((\mathfrak{n}I)R)^* :_R x)\\ &\subseteq ((\mathfrak{n}I :_A x)R)^* = (IR)^*,\end{aligned}$$

where the last containment is by Theorem 2.1. Thus $(IR)^*$ is $\mathfrak{m}$-full.

For (2), $(\mathfrak{n}I :_A \mathfrak{n}) = I$ and

$$\begin{aligned}(IR)^* &\subseteq (\mathfrak{m}(IR)^* :_R \mathfrak{m})\\ &\subseteq ((\mathfrak{n}R)^*(IR)^* :_R \mathfrak{n})\\ &\subseteq (((\mathfrak{n}I)R)^* :_R \mathfrak{n}R)\\ &\subseteq ((\mathfrak{n}I :_A \mathfrak{n})R)^* = (IR)^*,\end{aligned}$$

where the last containment is by Theorem 2.1. Thus $(IR)^*$ is basically full. □

Example 3.2. Let $R = k[[x,y,z]]/(x^2 - y^3 - z^7)$ and denote $\mathfrak{m} = (x,y,z)$. Note that y, z form a system of parameters and $\mathfrak{n} = (y,z)$ is the maximal ideal of $k[[y,z]]$. $(\mathfrak{n}R)^* = \mathfrak{m}$. Note that $(y,z)^n$ is $\mathfrak{n}$-full in $k[[y,z]]$. Thus, $((y,z)^nR)^*$ is $\mathfrak{m}$-full. Now by Proposition 2.4,

$$((y,z)^nR)^* = \bigcap_{1\le i\le n} ((y^i, z^{n-i+1})R)^* = (x)(y,z)^{n-1} + (y,z)^n$$

since $\mathfrak{m}$ is the test ideal of R and $xy^{i-1}z^{n-i}$ is the socle element of $((y^i, z^{n-i+1})R : \mathfrak{m})$.

$I = (y^n, y^{n-1}z^{n-1}, z^n)$ is basically-full in $k[[y,z]]$. Thus $(IR)^*$ is basically-full in R. Now by Proposition 2.4,

$$(IR)^* = ((y^n, z^{n-1})R)^* \cap (y^{n-1}, z^n)R)^* = (x(yz)^{n-2})(y,z) + I$$

since $xy(yz)^{n-2}$ is the socle element of $((y^n, z^{n-1})R : \mathfrak{m})$ and $xz(yz)^{n-2}$ is the socle element of $((y^{n-1}, z^n)R : \mathfrak{m})$.

Example 3.3. Let $R = k[[x,y]]/(xy)$. Note that $x^n + by^m$ is a system of parameters in R and $((x^n + by^m)R)^* = (x^n, y^m)$ [Va]. Note that we can't use $(x^n + by^m)$ in Theorem 3.1 unless $n = m = 1$. The only ideals of $k[[x + by]]$ are of the form $(x + by)^n$ and $(x + by)^nR = (x^n, y^n)$ and these ideals are $\mathfrak{m}$-full by Theorem 3.1. Note that (x^n, y^m) is $\mathfrak{m}$-full in R since $(\mathfrak{m}(x^n, y^m) : x + y) = (x^n, y^m)$, but we cannot obtain this from Theorem 3.1.

4. Tight Closure versions of $\mathfrak{m}$-full and basically full

Recall that integrally closed ideals are both $\mathfrak{m}$-full and basically full. By definition, I is $\mathfrak{m}$-full if there exists an $x \in \mathfrak{m} \setminus \mathfrak{m}^2$ such that $(\mathfrak{m}I : x) = I$. There may be an $x \in \mathfrak{m} \setminus \mathfrak{m}^2$ where $(\mathfrak{m}I : x)$ properly contains I even if I is integrally closed. Hence, for $\mathfrak{m}$-primary ideals if there exists an $x \in \mathfrak{m} \setminus \mathfrak{m}^2$ such that $(\mathfrak{m}\overline{I} : x) = \overline{I}$, we have the following chain: $I \subseteq (\mathfrak{m}I : \mathfrak{m}) \subseteq (\mathfrak{m}I : x) \subseteq (\mathfrak{m}\overline{I} : x) = \overline{I}$.

Also, the tight closure of an ideal always lies in the integral closure. If I is not $\mathfrak{m}$-full or basically-full, it may be that $(\mathfrak{m}I : x)$ or $(\mathfrak{m}I : \mathfrak{m})$ are contained in I^*. Hence, it makes sense to define tight closure versions of $\mathfrak{m}$-full and basically full. The containment above yields two different ways of defining tight closure versions of $\mathfrak{m}$-full and basically full. In some cases they will be equivalent, but they may not be in general. We illustrate with some examples in the next section.

Definition 4.1. *For an ideal I in a local ring $(R, \mathfrak{m})$, we say I is $*$-$\mathfrak{m}$-full if $(\mathfrak{m}I : x) = I^*$ for some $x \in \mathfrak{m} \setminus \mathfrak{m}^2$. We say that I is weakly $*$-$\mathfrak{m}$-full if $(\mathfrak{m}I : x)^* = I^*$ for some $x \in \mathfrak{m} \setminus \mathfrak{m}^2$.*

We will say that I is $*$-$\mathfrak{m}$-full (weakly $*$-$\mathfrak{m}$-full) with respect to x if $(\mathfrak{m}I : x) = I^*$ ($(\mathfrak{m}I : x)^* = I^*$).

Definition 4.2. *For an ideal I in a local ring $(R, \mathfrak{m})$, we say I is $*$-basically full if $(\mathfrak{m}I : \mathfrak{m}) = I^*$. We say that I is weakly $*$-basically full if $(\mathfrak{m}I : \mathfrak{m})^* = I^*$.*

Note that if I is $*$-$\mathfrak{m}$-full ($*$-basically full), then I is weakly $*$-$\mathfrak{m}$-full (weakly $*$-basically full). Also if I is $\mathfrak{m}$-full (basically full), I is weakly $*$-$\mathfrak{m}$-full ($*$-basically full). There are ideals I in a local Noetherian ring $(R, \mathfrak{m})$ where I is not $*$-$\mathfrak{m}$-full but is weakly $*$-$\mathfrak{m}$-full. We also have an example exhibiting that $*$-basically full is indeed a separate notion from weakly $*$-basically full. The following proposition sums up what is known.

Proposition 4.3. *Let (R, m) be a local Noetherian ring and I an ideal of R.*

(a) If I is weakly $*$-$\mathfrak{m}$-full with respect to x and $(\mathfrak{m}I : x)$ is tightly closed, then I is $*$-$\mathfrak{m}$-full.
(b) If I^* is $\mathfrak{m}$-full, then I is weakly $*$-$\mathfrak{m}$-full.
(c) If I is weakly $*$-basically full and $(\mathfrak{m}I : \mathfrak{m})$ is tightly closed, then I is $*$-basically full.
(d) If I^* is basically-full, then I is weakly $*$-basically full.
(e) Let $I \subseteq R$ be an ideal which is weakly $*$-$\mathfrak{m}$-full, then I is weakly $*$-basically full.

Proof: For (a), since $(\mathfrak{m}I : x)$ is tightly closed, then

$$(\mathfrak{m}I : x) = (\mathfrak{m}I : x)^* = I^*.$$

Hence I is $*$-$\mathfrak{m}$-full.

For (b), we know for some $x \in \mathfrak{m}$,

$$I \subseteq (\mathfrak{m}I : x) \subseteq (\mathfrak{m}I^* : x) = I^*.$$

Now take the tight closure of each ideal to observe

$$I^* \subseteq (\mathfrak{m}I : x)^* \subseteq (\mathfrak{m}I^* : x) = I^*.$$

As the ends are equal we see that there exists and $x \in \mathfrak{m}$ with $(\mathfrak{m}I : x)^* = I^*$. Hence, I is weakly $*$-$\mathfrak{m}$-full.

To see (c), since $(\mathfrak{m}I : \mathfrak{m})$ is tightly closed, then

$$(\mathfrak{m}I : \mathfrak{m}) = (\mathfrak{m}I : \mathfrak{m})^* = I^*.$$

Hence, I is $*$-basically full.

For (d),

$$I \subseteq (\mathfrak{m}I : \mathfrak{m}) \subseteq (\mathfrak{m}I^* : \mathfrak{m}) = I^*.$$

Now take the tight closure of each ideal to observe

$$I^* \subseteq (\mathfrak{m}I : \mathfrak{m})^* \subseteq (\mathfrak{m}I^* : \mathfrak{m}) = I^*.$$

As the ends are equal we see that $(\mathfrak{m}I : \mathfrak{m})^* = I^*$. Hence, I is weakly $*$-basically full.

Concluding with (e),

$$I \subseteq (\mathfrak{m}I : \mathfrak{m}) \subseteq (\mathfrak{m}I : x)^* = I^*.$$

Now take the tight closure of each ideal to observe

$$I^* \subseteq (\mathfrak{m}I : \mathfrak{m})^* \subseteq (\mathfrak{m}I : \mathfrak{m})^* = I^*.$$

As the ends are equal we see that $(\mathfrak{m}I : \mathfrak{m})^* = I^*$. Hence, I is weakly $*$-basically full. □

As the Rees Property was related to $\mathfrak{m}$-fullness, we would like to define a tight closure notion of Rees property too. Recall, $\mu(I) = \dim_{R/\mathfrak{m}}(I/\mathfrak{m}I)$. If $J \subseteq I$, we know that $J^* \subseteq I^*$. The natural definition of the $*$-Rees Property is the following.

Definition 4.4. *We say I satisfies the $*$-Rees Property if for all $J \supseteq I$, $\mu(J^*) \leq \mu(I^*)$.*

Proposition 4.5. *Let $I \subseteq R$ be an ideal satisfying I^* satisfies the Rees property, then I satisfies the $*$-Rees property.*

Proof: If I^* satisfies the Rees property, then for every $J \supseteq I^*$, $\mu(J) \leq \mu(I^*)$. For all $J \supseteq I$, $J^* \supseteq I^*$. Thus $\mu(J^*) \leq \mu(I^*)$. Hence, I satisfies the $*$-Rees property. □

In the case that $(R, \mathfrak{m})$ is a local normal isolated singularity with test ideal equal to $\mathfrak{m}$, Hara and Smith [HS] have shown that $\mathfrak{m}$ is a strong test ideal. In other words, $\mathfrak{m}I = \mathfrak{m}I^*$ for all $I \subseteq R$. This is equivalent to $I^* \subseteq (\mathfrak{m}I : \mathfrak{m})$. We use this containment to show the following.

Proposition 4.6. *Let $(R, \mathfrak{m})$ be a Noetherian local ring with test ideal equal to $\mathfrak{m}$ and $I \subseteq R$ an ideal.*

(a) I is $*$-$\mathfrak{m}$-full if and only if I is weakly $*$-$\mathfrak{m}$-full.
(b) I $*$-basically full if and only if I is weakly $*$-basically full.

Proof: To see (a), since $\mathfrak{m}$ is a strong test ideal, $I^* \subseteq (\mathfrak{m}I : \mathfrak{m}) \subseteq (\mathfrak{m}I : x) \subseteq (\mathfrak{m}I : x)^* = I^*$. Hence, $(\mathfrak{m}I : x) = I^*$ which implies I is $*$-$\mathfrak{m}$-full.

To see (b), note that if I is $*$-basically full, then I is weakly $*$-basically full since $(\mathfrak{m}I : \mathfrak{m}) = I^*$ is tightly closed, hence $(\mathfrak{m}I : \mathfrak{m})^* = I^*$. When $\mathfrak{m}$ is the test ideal of R, $\mathfrak{m}$ is a strong test ideal. Hence $I^* \subseteq (\mathfrak{m}I : \mathfrak{m})$. If I is weakly $*$-basically full then $I^* \subseteq (\mathfrak{m}I : \mathfrak{m}) \subseteq (\mathfrak{m}I : \mathfrak{m})^* = I^*$. Hence, we have equality throughout and $(\mathfrak{m}I : \mathfrak{m}) = I^*$ and I is $*$-basically full. □

Proposition 4.7. *Let $(R, \mathfrak{m})$ be a local normal isolated singularity with test ideal equal to $\mathfrak{m}$ and $I \subseteq R$ an ideal of R which is $*$-$\mathfrak{m}$-full. The following hold.*

(a) I^* is $\mathfrak{m}$-full.
(b) I is $*$-basically full.
(c) I^* is basically full.
(d) I^* is full.
(e) I satisfies the $*$-Rees Property.
(f) Now suppose that I is $\mathfrak{m}$-primary, then $\mu(I^*) = \lambda(R/I + xR) + \mu(I + xR/xR)$.

Proof: To see (a)-(c), observe that the following inclusions:

$$I^* \subseteq (\mathfrak{m}I : \mathfrak{m}) = (\mathfrak{m}I^* : \mathfrak{m}) \subseteq (\mathfrak{m}I^* : x) = (\mathfrak{m}I : x) = I^*$$

become equalities. Hence, $(\mathfrak{m}I^* : \mathfrak{m}) = I^*$ or I^* is $\mathfrak{m}$-full, $(\mathfrak{m}I : \mathfrak{m}) = I^*$ or I is $*$-basically full and $(\mathfrak{m}I^* : x) = I^*$ or I^* is basically full.

For (d), note

$$(I^* : x) \subseteq ((\mathfrak{m}I : \mathfrak{m}) : x) = ((\mathfrak{m}I : x) : \mathfrak{m}) = (I^* : \mathfrak{m}) \subseteq (I^* : x).$$

Hence, I^* is full.

For (e), using the fact that I^* is $\mathfrak{m}$-full, then I^* satisfies the Rees Property. So for every $J \supseteq I^*$, $\mu(J) \leq \mu(I^*)$. Note that if $J \supseteq I$, then $J^* \supseteq I^*$. Thus $\mu(J^*) \leq \mu(I^*)$.

For (f), we apply [Go, Lemma 2.2] to obtain $\mu(I^*) = \lambda((\mathfrak{m}I : x)/\mathfrak{m}I) = \lambda(R/I + xR) + \mu(I + xR/xR)$.

□

Note that if I is $*$-$\mathfrak{m}$-full, then $I^* = (\mathfrak{m}I : x)$ for some $x \in \mathfrak{m} \setminus \mathfrak{m}^2$. Also if τ is a strong test ideal, we have the following containments: $I^* \subseteq (\tau I : \tau) \subseteq (\mathfrak{m}I : \tau) \subseteq (\mathfrak{m}I : y)$ for some $y \in \tau \setminus \tau^2$. If $y \in \mathfrak{m} \setminus \mathfrak{m}^2$, then this implies that $(\tau I : \tau) = I^*$.

5. Examples

To show that the concepts of $*$-$\mathfrak{m}$-fullness and $*$-basically fullness are new we include several examples. First we give some examples of $*$-$\mathfrak{m}$-full ideals which are not $\mathfrak{m}$-full.

Example 5.1. Let $R = k[[x, y, z]]/(x^2 - y^3 - z^7)$, $\mathrm{char}(k) = p > 7$ and $I = (y, z)$. Note that as $\mathfrak{m}$ is the test ideal and y, z form a system of parameters

$$((y, z) : \mathfrak{m}) = (x, y, z) = (y, z)^*$$

by Proposition 2.2 and

$$(\mathfrak{m}(y, z) : x) = ((xy, xz, y^2, yz, z^2) : x) = (x, y, z) = (y, z)^* \neq (y, z).$$

So I is not $\mathfrak{m}$-full, but I is $*$-$\mathfrak{m}$-full. Also $(y, z)^n$ as in Example 3.2 is also $*$-$\mathfrak{m}$-full, but not $\mathfrak{m}$-full.

Note that (y, z) in Example 5.1 satisfies the $*$-Rees Property as $(y, z)^* = (x, y, z)$ and the only ideal containing (y, z) is $\mathfrak{m}$ which is three generated.

Example 5.2. Let $R = k[[x, y, z]]/(x^2 - y^3 - z^6)$, $\mathrm{char}(k) = p > 5$ and $I = (y^2, yz, z^3)$. Note that

$$(y^2, yz, z^3)^* = ((y^2, yz, z^3) : \mathfrak{m}) = (xy, xz^2, y^2, yz, z^3)$$

and

$$\begin{aligned}(\mathfrak{m}(y^2, yz, z^3) : x + z) &= ((xy^2, xyz, xz^3, y^3, y^2z, yz^2, z^4) : x + z) \\ &= (xy, xz^2, y^2, yz, z^3) \neq (y^2, yz, z^3).\end{aligned}$$

So I is not $\mathfrak{m}$-full, but I is $*$-$\mathfrak{m}$-full.

The next example offers an ideal which is weakly $*$-$\mathfrak{m}$-full but not $\mathfrak{m}$-full or $*$-$\mathfrak{m}$-full.

Example 5.3. Let $R = k[[t^3, t^5]]$. Since $(\mathfrak{m}(t^8) : t^3) = (t^8, t^{10})$, $(\mathfrak{m}(t^8) : t^5) = (t^6, t^8)$, and for $a \neq 0$, $(\mathfrak{m}(t^8) : t^3 + at^5) = (t^8, t^{10} - at^{12})$ and $(\mathfrak{m}(t^8) : t^5 + at^6) = (t^8, t^9 - at^{10})$ which are all not tightly closed. Hence, (t^8) is not $\mathfrak{m}$-full or $*$-$\mathfrak{m}$-full. However, $(\mathfrak{m}(t^8) : t^3)^* = (t^8, t^9, t^{10})$ which is tightly closed. Hence, (t^8) is weakly $*$-$\mathfrak{m}$-full.

The following two examples exhibit that there are ideals which are $*$-$\mathfrak{m}$-full and weakly $*$-basically full, but not $*$-basically full.

Example 5.4. Let $R = k[[x, y, z]]/(x^2 - y^3 - z^{12})$, $\mathrm{char}(k) = p > 11$ and $I = (y, z^2)$. The test ideal of R is (x, y, z^2); hence, $(y, z^2)^* = ((y, z^2) : (x, y, z^2)) = (x, y, z^2)$. The ideal (y, z^2) is $*$-$\mathfrak{m}$-full $(\mathfrak{m}(y, z^2) : x) = (x, y, z^2)$. However, (y, z^2) is not basically full nor $*$-basically full, since $(\mathfrak{m}(y, z^2) : \mathfrak{m}) = (xz, y, z^2)$ is not equal to (y, z^2), nor $(y, z^2)^*$. However, (y, z^2) is weakly $*$-basically full.

Example 5.5. Let $R = k[[t^2, t^5]]$. $(\mathfrak{m}(t^4) : \mathfrak{m}) = (t^4, t^7)$ which is not tightly closed. Hence, (t^4) is not basically full or $*$-basically full. However, $(\mathfrak{m}(t^4) : \mathfrak{m})^* = (t^4, t^5)$ which is tightly closed. Hence, (t^4) is weakly $*$-basically full. Note, (t^4) is $*$-$\mathfrak{m}$-full since $(\mathfrak{m}(t^4) : t^4 + t^5) = (t^4, t^5)$. This gives an example of an ideal which is $*$-$\mathfrak{m}$-full but not $*$-basically full.

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