

A FORMULA FOR THE *-CORE OF AN IDEAL

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ABSTRACT. Expanding on the 2010 work of Fouli and Vassilev, we determine a formula for the $*$ -core of an ideal in two different settings: (1) in a Cohen–Macaulay local ring of characteristic $p > 0$, with perfect residue field and test ideal of depth at least two, where the ideal has a minimal $*$ -reduction that is a parameter ideal, and (2) in a normal local domain of characteristic $p > 0$, with perfect residue field and \mathfrak{m} -primary test ideal, where the ideal is a sufficiently high Frobenius power of an ideal. We also exhibit some examples where our formula fails if our hypotheses are not met.

1. INTRODUCTION

In a recent paper [FV], Fouli and Vassilev introduced the cl -core of an ideal for a closure operation cl . Like the core of an ideal originally defined by Rees and Sally [RS], the cl -core of an ideal is the intersection of all the cl -reductions of the ideal, where J is a cl -reduction of an ideal I if $J \subseteq I$ and $J^{cl} = I^{cl}$. Of particular interest for rings of characteristic $p > 0$ is the $*$ -core, where $*$ denotes tight closure.

The tight closure of an ideal is contained in the integral closure but is generally much closer to the ideal than the integral closure. Note that a $*$ -reduction of an ideal I is also a reduction. Hence, $\text{core}(I) \subseteq * \text{-core}(I)$. Recall that $\text{core}(I)$ sits deep within I ; $* \text{-core}(I)$ will lie within I but will be generally much closer to I . In [FV], Fouli and Vassilev determined that the $*$ -core and the core of an ideal I agree if the analytic spread of I is equal to the $*$ -spread of I , where the $*$ -spread is the minimal number of generators of any minimal $*$ -reduction of I . Fouli and Vassilev also exhibited some examples where the core and the $*$ -core of an ideal are not equal.

Huneke and Swanson [HS1] determined that in a 2-dimensional regular local ring the core of an integrally closed ideal I is given by $\text{core}(I) = (J : I)I$, where J is a minimal reduction of I . This result was later generalized by Corso, Polini and Ulrich [CPU2] under suitable assumptions on I (not necessarily integrally closed) in the case when (R, \mathfrak{m}) is a Gorenstein local ring. In particular, if I is an \mathfrak{m} -primary ideal with reduction number at most 1, then this formula holds [CPU2, Theorem 2.6]. We will discuss the additional assumptions on I in Section 2.

Various authors have developed techniques for studying the core of an ideal (see for example [CPU1], [CPU2], [HT], [PU], [HySm1], [HySm2]). However, these

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techniques do not apply when studying the $*$ -core and minimal $*$ -reductions. For one, the analytic spread of an ideal $\ell(I)$ is always bounded above by the dimension of the ring and the $*$ -spread can grow arbitrarily large. In order to better understand minimal $*$ -reductions and the $*$ -core of an ideal we make use of special tight closure techniques introduced by Vraciu [Vr1] and further developed in [HV], [Ep], and [EV]. To work with special tight closure, we need to assume that R is an excellent normal local ring of characteristic $p > 0$ with perfect residue field. Using techniques of special tight closure we obtain a formula for the $*$ -core of an ideal that is very similar to the formula shown by Corso, Polini and Ulrich in [CPU2, Theorem 2.6]; however, with minimal $*$ -reductions instead of minimal reductions. The ideals we consider in this paper have reduction number equal to one with respect to their minimal $*$ -reductions.

The outline of the paper is as follows: In Section 2, we briefly introduce the reader to tight closure and conditions associated to the study of the core. In Section 3, over an excellent normal local ring, we provide two criteria that when satisfied establish that $*$ -core(I) = $(J : I)J = (J : I)I$, where J is any minimal $*$ -reduction of an ideal I (Theorem 3.4). We subsequently use these criteria to show that when R is a Cohen–Macaulay, normal, excellent local ring of characteristic $p > 0$ with perfect residue field and test ideal of depth at least two, then $*$ -core(I) = $(J : I)J = (J : I)I$, where I is an ideal that has a minimal $*$ -reduction J generated by part of a system of parameters (Theorem 3.7). If (R, \mathfrak{m}) is a normal local ring of characteristic $p > 0$ and test ideal equal to \mathfrak{m} , we show that $*$ -core(I) = $(J : I)J = (J : I)I$, where I is an ideal and J is any minimal $*$ -reduction of I (Theorem 3.10). Furthermore, when (R, \mathfrak{m}) is a normal local ring of characteristic $p > 0$, with perfect residue field and \mathfrak{m} -primary test ideal, we determine that $*$ -core($I^{[q]}$) = $(I^{[q]} : J^{[q]})J^{[q]}$ for q a sufficiently large power of p and for every minimal $*$ -reduction J of I (Theorem 3.12). In Section 4, we exhibit two examples where our formula fails, suggesting that if the criteria of Theorem 3.4 are not satisfied, then the conclusion need not hold.

2. PRELIMINARIES

Since the $*$ -core of an ideal I is based on reductions stemming from the tight closure of I , we will review some tight closure concepts to help clarify these notions for the reader. For a more extensive introduction, see [Hu] or [HH1].

Definition 2.1. Let R be a Noetherian ring of characteristic $p > 0$. We denote positive powers of p by q and the set of elements of R which are not contained in the union of minimal primes by R^0 . Then

- (a) For any ideal $I \subseteq R$, $I^{[q]}$ is the ideal generated by the q th powers of elements in I .
- (b) We say an element $x \in R$ is in the *tight closure*, I^* , of I if there exists a $c \in R^0$ such that $cx^q \in I^{[q]}$ for all large q .

Finding the tight closure of an ideal would be hard without test elements and test ideals. A *test element* is an element $c \in R^0$ such that $cI^* \subseteq I$ for all $I \subseteq R$. Note that $c \in \bigcap_{I \subseteq R} (I : I^*)$. The ideal $\tau = \bigcap_{I \subseteq R} (I : I^*)$ is called the *test ideal* of R .

In a Noetherian local ring of characteristic $p > 0$, Vraciu [Vr1] defined the special tight closure, I^{*sp} , to be the elements $x \in R$ such that $x^{q_0} \in (\mathfrak{m}I^{[q_0]})^*$ for some

$q_0 = p^{e_0}$. Vraciu shows that $I^{*sp} \cap I = \mathfrak{m}I$ if I is generated by *-independent elements [Vr1, Proposition 4.2]. Furthermore, Huneke and Vraciu show in [HV, Theorem 2.1] that $I^* = I + I^{*sp}$, without the assumption that I is generated by *-independent elements. Recall that the elements f_1, \dots, f_n are said to be *-independent if $f_i \notin (f_1, \dots, \hat{f}_i, \dots, f_n)^*$ for all $i = 1, \dots, n$.

Note that the minimal *-reductions of I are generated by *-independent elements. In particular, Epstein shows that if R is an excellent analytically irreducible local domain, then every minimal *-reduction is generated by the same minimal number of generators, namely the *-spread of I [Ep, Theorem 5.1]. He also showed that $I^{*sp} = J^{*sp}$ for all *-reductions J of I [Ep, Lemma 3.4].

We mention here the following result of Aberbach that is used in many arguments in this paper.

Proposition 2.2 ([Ab, Proposition 2.4]). *Let (R, \mathfrak{m}) be an excellent, analytically irreducible local ring of characteristic $p > 0$, let I be an ideal, and let $f \in R$. Assume that $f \notin I^*$. Then there exists $q_0 = p^{e_0}$ such that for all $q \geq q_0$ we have $I^{[q]} : f^q \subseteq \mathfrak{m}^{[q/q_0]}$.*

We conclude the section by recalling some definitions and conditions associated with the study of the core. We also provide some instances for when these conditions are satisfied.

Let (R, \mathfrak{m}) be a Noetherian local ring and let I be an ideal in R . Let $d = \dim R$.

- (a) The *analytic spread* $\ell = \ell(I)$ is the dimension of the special fiber ring of I , $\mathcal{F}(I) = R[It]/\mathfrak{m}R[It] = R/\mathfrak{m} \oplus I/\mathfrak{m}I \oplus I^2/\mathfrak{m}I^2 \oplus \dots$. It is well known that $\ell(I) = \mu(J)$ for every minimal reduction J of I , where $\mu(J)$ denotes the minimal number of generators of J [NR].
- (b) I satisfies G_s if $\mu(I_{\mathfrak{p}}) \leq s - 1$ for every prime $\mathfrak{p} \supseteq I$, with $\dim R_{\mathfrak{p}} \leq s - 1$. If I is \mathfrak{m} -primary, then I satisfies G_d automatically. More generally, if I is equimultiple, i.e. $\ell = \text{ht } I$, then I satisfies G_{ℓ} . This condition is rather mild and there are many classes of ideals that would satisfy this condition (see for example [CPU1], [CPU2]).
- (c) K is a geometric s -residual intersection of an ideal I if there exists an s -generated ideal \mathfrak{a} with $K = \mathfrak{a} : I$ and $\text{ht } K \geq s \geq \text{ht } I = g$ and $\text{ht } (I + K) \geq s + 1$. Residual intersections are a generalization of linkage theory when R is a local Gorenstein ring or I is an unmixed ideal. In this case if $s = g$, then for K to be a g -residual intersection of I is equivalent to K being linked to I since the ideal \mathfrak{a} will be a complete intersection. This topic has been studied in great detail by various authors (see [CEU], [JU], [U]).
- (d) I is weakly s -residually S_2 if R/K satisfies Serre's condition S_2 for every geometric i -residual intersection with $i \leq s$. Again we remark here that if I is \mathfrak{m} -primary, or more generally equimultiple, then I is weakly $(\ell - 1)$ -residually S_2 .

3. CRITERIA AND FORMULAS FOR THE *-CORE

The following observation is an immediate consequence of the definition of the test ideal.

Observation 3.1. Let R be a ring of characteristic $p > 0$, with test ideal τ . Let I be an ideal. Then $\tau I \subseteq \text{-core}(I)$.

Proof. Let J be an arbitrary $*$ -reduction of I . Since $I \subseteq J^*$, it follows that $\tau \subseteq (J : I)$, and thus $\tau I \subseteq J$. \square

We will use the following characterization of minimal $*$ -reductions from [Vr2].

Theorem 3.2 ([Vr2, Theorem 1.13]). *Let (R, \mathfrak{m}) be an excellent normal local ring of characteristic $p > 0$ and perfect residue field. Let $K \subseteq I$ be ideals, such that K is generated by n elements, where n is the $*$ -spread of I . Assume that I is tightly closed. The following are equivalent:*

- (a) K is a minimal $*$ -reduction of I .
- (b) $I^{*sp} \cap K \subseteq \mathfrak{m}I$.
- (c) $I = I^{*sp} + K$.

Using Theorem 3.2 we can describe explicitly all minimal $*$ -reductions of an ideal.

Lemma 3.3. *Let (R, \mathfrak{m}) be an excellent, normal local ring of characteristic $p > 0$ and perfect residue field. Let I be an ideal and let J be a minimal $*$ -reduction of I . Then we may write $I = J + (u_1, \dots, u_s)$, with $u_1, \dots, u_s \in J^{*sp}$. Suppose that f_1, \dots, f_n is a minimal system of generators of J and let $K \subseteq I$. Then K is a minimal $*$ -reduction of I if and only if $K = (f_1 + v_1, f_2 + v_2, \dots, f_n + v_n)$, with $v_1, \dots, v_n \in (u_1, \dots, u_s)$.*

Proof. First notice that J is a minimal $*$ -reduction of I if and only if J is a minimal $*$ -reduction of I^* and thus we may use Theorem 3.2 applied to I^* . Furthermore, any minimal generating set of J can be extended to a minimal generating set for I by [Ep, Lemma 2.2]. Notice that $J \subseteq I \subseteq I^* = J + J^{*sp}$ by Theorem 3.2 and [Ep, Lemma 3.4]. Hence we may write $I = J + (u_1, \dots, u_s)$, where u_1, \dots, u_s may be chosen from J^{*sp} .

We check that for every $K = (f_1 + v_1, \dots, f_n + v_n)$ as in the statement, condition (b) in Theorem 3.2 holds. Indeed, if $c_1(f_1 + v_1) + \dots + c_n(f_n + v_n) \in I^{*sp}$, then we must have $c_1 f_1 + \dots + c_n f_n \in I^{*sp} \cap J \subseteq \mathfrak{m}I$, where the last inclusion follows from [Vr2, Observation 1.11], noting that J is a minimal $*$ -reduction of I . Since $\sum_{i=1}^n c_i f_i \in \mathfrak{m}I$ and $\{f_1, \dots, f_n\}$ is a minimal generating set for J , then $c_i \in \mathfrak{m}$ for all $i = 1, \dots, n$ and thus $I^{*sp} \cap K \subseteq \mathfrak{m}I$.

Conversely, let $K \subseteq I$ be a minimal $*$ -reduction. Since J is a minimal $*$ -reduction of I , we know that the $*$ -spread of I is n , the minimal number of generators of J . Thus, K must also be minimally generated by n elements, and we can write $K = (g_1 + v_1, \dots, g_n + v_n)$, with $g_i \in (f_1, \dots, f_n)$, and $v_i \in (u_1, \dots, u_s)$ for all $i = 1, \dots, n$.

We claim that we must have $(g_1, \dots, g_n) = (f_1, \dots, f_n)$. Assuming this claim, we can write $\underline{f} = A\underline{g}$, with A an invertible $n \times n$ matrix with entries in R , where \underline{f} denotes the vector $(f_1, \dots, f_n)^T$. The entries of $A(g_1 + v_1, \dots, g_n + v_n)^T$ are then generators for K , and they can be written in the form $f_1 + v'_1, \dots, f_n + v'_n$, where $\underline{v}' = A\underline{v}$.

Assuming by contradiction that the claim is not true, we can choose a minimal system of generators f'_1, \dots, f'_n for J such that $(g_1, \dots, g_n) \subseteq \mathfrak{m}J + (f'_2, \dots, f'_n)$. In order to achieve this, one can take f'_1, \dots, f'_n to be the pull-backs to R of a vector space basis $\overline{f'_1}, \dots, \overline{f'_n}$ of $J/\mathfrak{m}J$ obtained by extending a basis of the subspace spanned by $\overline{g_1}, \dots, \overline{g_n}$.

Since K is a minimal $*$ -reduction of I , we must have $f'_1 \in K^*$, and therefore $f'_1 \in (\mathfrak{m}f'_1, f'_2, \dots, f'_n, u_1, \dots, u_s)^*$. It follows that $f'_1 \in (f'_2, \dots, f'_n, u_1, \dots, u_s)^*$ by the tight closure version of Nakayama's Lemma in [Ep, Proposition 2.1]. Since $u_1, \dots, u_s \in J^{*sp}$, we see that in fact we must have $f'_1 \in (f'_2, \dots, f'_n)^*$. Indeed, there exists $c \in R^0$ such that

$$cf_1^{q_0} = a_2 f_2^{q_0} + \dots + a_n f_n^{q_0} + b_1 u_1^{q_0} + \dots + b_s u_s^{q_0},$$

for some $a_i, b_j \in R$ with $2 \leq i \leq n$ and $1 \leq j \leq s$. Since $u_j \in J^{*sp}$ for all j there exists q_0 such that we can rewrite the above equation to obtain:

$$\begin{aligned} c^2 f_1^{q_0} &= ca_2 f_2^{q_0} + \dots + ca_n f_n^{q_0} + cb_1 u_1^{q_0} + \dots + cb_s u_s^{q_0} \\ &= a'_1 f_1^{q_0} + (ca_2 + a'_2) f_2^{q_0} + \dots + (ca_n + a'_n) f_n^{q_0}, \end{aligned}$$

with $a'_1, \dots, a'_n \in \mathfrak{m}^{q_0}$. The conclusion follows by applying Proposition 2.2. This contradicts the fact that f'_1, \dots, f'_n are $*$ -independent. \square

Theorem 3.4. *Let (R, \mathfrak{m}) be a normal local ring of characteristic $p > 0$ with perfect residue field. Let I be an ideal and let J be a minimal $*$ -reduction of I . Then we may write $I = J + (u_1, \dots, u_s)$, with $u_1, \dots, u_s \in J^{*sp}$. Suppose that f_1, \dots, f_n is a minimal system of generators of J .*

- (a) *If $(f_1, \dots, \hat{f}_i, \dots, f_n) : f_i \subseteq (J : I)$ for all $i = 1, \dots, n$, and $u_j(J : u_j) \subseteq (J : I)J$ for all $j = 1, \dots, s$, then $*\text{-core}(I) \subseteq (J : I)J$.*
- (b) *If $u_j(J : I) \subseteq \mathfrak{m}(J : I)J$ for all $j = 1, \dots, s$, then $*\text{-core}(I) \supseteq (J : I)J$.*
- (c) *If the assumptions of both part (a) and part (b) are satisfied, then*

$$*\text{-core}(I) = (J : I)J = (J : I)I.$$

Proof. (a) Let $A_1 f_1 + \dots + A_n f_n \in *\text{-core}(I)$. Fix $i \in \{1, \dots, n\}$ and let $j \in \{1, \dots, s\}$. We wish to show that $A_i u_j \in J$. Since j is arbitrary, it will follow that $A_i \in (J : I)$.

Let $J' = (f_1, \dots, f_{i-1}, f_i + u_j, f_{i+1}, \dots, f_n)$ be another minimal $*$ -reduction of I . We must have $A_1 f_1 + \dots + A_n f_n = B_1 f_1 + \dots + B_{i-1} f_{i-1} + B_i (f_i + u_j) + \dots + B_n f_n$ for some $B_1, \dots, B_n \in R$. Note we must have $B_i u_j \in u_j(J : u_j) \subseteq (J : I)J$ by assumption, and thus we can write $B_i u_j = C_1 f_1 + \dots + C_n f_n$ with $C_1, \dots, C_n \in (J : I)$. Therefore $A_i - B_i - C_i \in (f_1, \dots, \hat{f}_i, \dots, f_n) : f_i \subseteq (J : I)$ and, since $C_i \in (J : I)$, then $A_i - B_i \in (J : I)$. Multiplying by u_j we obtain $A_i u_j - B_i u_j \in (J : I)I$ and thus $A_i u_j \in (J : I)I \subseteq J$.

(b) Let $A_1 f_1 + \dots + A_n f_n \in (J : I)J$, with $A_1, \dots, A_n \in (J : I)$, and let J' be another minimal $*$ -reduction of I . Then by Lemma 3.3 we can assume that $J' = (f_1 + v_1, \dots, f_n + v_n)$, where $v_1, \dots, v_n \in (u_1, \dots, u_s)$. We wish to find $B_1, \dots, B_n \in (J : I)$ such that

$$(1) \quad A_1 f_1 + \dots + A_n f_n = B_1 (f_1 + v_1) + \dots + B_n (f_n + v_n).$$

Write $(J : I) = (g_1, \dots, g_l)$, and $A_i = \sum_{j=1}^l \alpha_{ij} g_j$, $B_i = \sum_{j=1}^l y_{ij} g_j$, where y_{ij} are

unknowns. By assumption, we can write $v_i g_k = \sum_{\nu=1}^n \sum_{j=1}^l \gamma_{ikj\nu} g_j f_\nu$, with $\gamma_{ikj\nu} \in \mathfrak{m}$ for all $i = 1, \dots, n$ and all $k = 1, \dots, l$.

A sufficient condition for equation (1) to hold is that the coefficients of f_ν on each side of the equation are the same for all $\nu = 1, \dots, n$, which leads to the system

of linear equations

$$(2) \quad \sum_{j=1}^l y_{\nu j} g_j + \sum_{j=1}^l \sum_{i=1}^n \sum_{k=1}^l y_{ik} \gamma_{ikj\nu} g_j = \sum_{j=1}^l \alpha_{\nu j} g_j \quad \text{for all } \nu = 1, \dots, n.$$

A further sufficient condition is obtained by identifying the coefficients of each g_j on the two sides of the equations (2). The following system of linear equations is thus obtained:

$$(3) \quad (\nu, j) : \quad y_{\nu j} + \sum_{i=1}^n \sum_{k=1}^l y_{ik} \gamma_{ikj\nu} = \alpha_{\nu j} \quad \text{for all } \nu = 1, \dots, n, \quad j = 1, \dots, l.$$

This is a system of nl equations with nl unknowns. Note that each $y_{\nu j}$ appears with a unit coefficient in equation (ν, j) and with a coefficient in \mathfrak{m} in all the other equations, and therefore the matrix of coefficients can be row-reduced to the identity matrix, showing that there exists a unique solution for the system. \square

Corollary 3.5. *With the same setup as in Theorem 3.4 if the assumptions of both part (a) and part (b) are satisfied, $*\text{-core}(I)$ is obtained as a finite intersection of minimal $*\text{-reductions}$, namely*

$$*\text{-core}(I) = J \cap \left(\bigcap_{i,j} J_{i,j} \right),$$

where $J = (f_1, \dots, f_n)$ and $J_{i,j} := (f_1, \dots, f_{i-1}, f_i + u_j, f_{i+1}, \dots, f_n)$, for all $i \in \{1, \dots, n\}$ and $j \in \{1, \dots, s\}$.

Proof. The proof of part (a) of Theorem 3.4 shows that $J \cap \left(\bigcap_{i,j} J_{i,j} \right) \subseteq (J : I)J$, while part (b) of Theorem 3.4 shows that $(J : I)J \subseteq *\text{-core}(I)$. The conclusion follows, since $*\text{-core}(I) \subseteq J \cap \left(\bigcap_{i,j} J_{i,j} \right)$ by the definition of the $*\text{-core}$. \square

Remark 3.6. Let (R, \mathfrak{m}) be a Cohen–Macaulay, excellent, normal local ring with $\dim R \geq 2$, characteristic $p > 0$ and perfect residue field. Let τ be the test ideal of R . The normality assumption implies that $\text{depth } \tau \geq 2$. This is because the defining ideal of the non-regular locus has height at least two, and [HH2, Theorem 6.2] shows that we can pick a regular sequence consisting of two elements c, d in this ideal and replace them by sufficiently large powers in order to obtain a regular sequence of length two consisting of test elements. Thus we may assume that one can choose a system of parameters x_1, \dots, x_d for R with $x_1, x_2 \in \tau$.

Theorem 3.7. *Let (R, \mathfrak{m}) be a Cohen–Macaulay, excellent, normal local ring with $\dim R \geq 2$, characteristic $p > 0$, and perfect residue field. Let x_1, \dots, x_d be part of a system of parameters, with $x_1, x_2 \in \tau$, where τ is the test ideal of R . Let I be an ideal and suppose $J = (x_1^t, x_2^t, x_3, \dots, x_d) \subseteq I \subseteq J^*$, where t is a positive integer. If $t \geq 2$, then $*\text{-core}(I) \subseteq (J : I)J = (J : I)I$; and when $t \geq 3$, then equality is obtained.*

Proof. Let $I = J + (u_1, \dots, u_s)$, where $u_i \in I^{*sp}$ for all $1 \leq i \leq s$. We need to see that all the hypotheses listed in part (a) and part (b) of Theorem 3.4 hold.

To confirm that the hypotheses of part (a) hold, notice that

$$(x_1^t, x_2^t, x_3, \dots, \widehat{x}_i, \dots, x_d) : x_i = (x_1^t, x_2^t, x_3, \dots, \widehat{x}_i, \dots, x_d) \subseteq J \subseteq (J : I).$$

For the second hypothesis in part (a), note that

$$J^* \subseteq J : \tau \subseteq J : (x_1, x_2) = (x_1^t, x_2^t, x_3, \dots, x_d, (x_1 x_2)^{t-1}).$$

We claim that $J^{*sp} \subseteq \mathfrak{m}J + (x_1 x_2)^{t-1}$. To verify this, suppose that $f \in J^{*sp}$. Since $J^* = J + J^{*sp} \subseteq J + (x_1 x_2)^{t-1}$, we may write $f = g + u(x_1 x_2)^{t-1}$ for some $g \in J$ and $u \in R$. Since $f, g \in J^*$, then $u(x_1 x_2)^{t-1} \in J^*$. Then there exists $c \in R^0$ such that for all large powers q of p we have $cu^q(x_1 x_2)^{q(t-1)} \in J^{[q]} = (x_1^{qt}, x_2^{qt}, x_3^q, \dots, x_d^q)$. Therefore

$$cu^q \in (x_1^{qt}, x_2^{qt}, x_3^q, \dots, x_d^q) : (x_1 x_2)^{q(t-1)} = (x_1^q, x_2^q, x_3^q, \dots, x_d^q).$$

Hence

$$\begin{aligned} cu^q(x_1 x_2)^{q(t-1)} &\in (x_1 x_2)^{q(t-1)}(x_1^q, x_2^q, x_3^q, \dots, x_d^q) \\ &= (x_1^{qt} x_2^{q(t-1)}, x_2^{qt} x_1^{q(t-1)}, (x_1 x_2)^{q(t-1)} x_3^q, \dots, (x_1 x_2)^{q(t-1)} x_d^q) \\ &\subseteq (x_1^q, x_2^q)(x_1^{qt}, x_2^{qt}, x_3^q, \dots, x_d^q) = (x_1^q, x_2^q)J^{[q]} \subseteq \mathfrak{m}^{[q]}J^{[q]}, \end{aligned}$$

where the penultimate inclusion holds as long as $t \geq 2$. Therefore $u(x_1 x_2)^{(t-1)} \in (\mathfrak{m}J)^*$, and by definition of the special tight closure we have $u(x_1 x_2)^{(t-1)} \in J^{*sp}$. As $f \in J^{*sp}$, then $g \in J^{*sp}$ as well. But $g \in J$ also and thus $g \in J \cap J^{*sp} = \mathfrak{m}J$, by [Vr1, Proposition 4.2]. Therefore $J^{*sp} \subseteq \mathfrak{m}J + (x_1 x_2)^{t-1}$. Since the hypotheses in Theorem 3.4 are unaffected when the u_i 's are replaced by $u_i + w_i$ with $w_i \in \mathfrak{m}J$, we may assume without loss of generality that we can write $u_j = (x_1 x_2)^{t-1} u'_j$.

We have

$$u_j(J : u_j) = (x_1 x_2)^{t-1} u'_j((x_1, x_2, x_3, \dots, x_d) : u'_j) \subseteq (x_1, x_2)J \subseteq (J : I)J.$$

The last inclusion follows from the fact that $(x_1, x_2) \subseteq \tau \subseteq (J : I)$.

Finally we establish that the hypothesis in part (b) holds. As above, we can write $u_j = (x_1 x_2)^{t-1} u'_j$, and thus $(J : I) \subseteq (x_1, x_2, x_3, \dots, x_d) : (u'_1, \dots, u'_s)$. Therefore

$$u_j(J : I) \subseteq (x_1 x_2)^{t-1}(x_1, x_2, x_3, \dots, x_d) \subseteq (x_1^{t-1}, x_2^{t-1})J \subseteq \mathfrak{m}J(J : I),$$

where the last inclusion follows from the fact that $(x_1^{t-1}, x_2^{t-1}) \subseteq \mathfrak{m}(x_1, x_2) \subseteq \mathfrak{m}(J : I)$ when we take $t \geq 3$. \square

The following observation can be obtained as a corollary of Theorem 3.7 and [FV, Theorem 4.8], using [CPU2, Theorem 2.6]. Nonetheless, we give an elementary direct proof.

Observation 3.8. Let R be a Cohen–Macaulay, excellent, normal local ring with $\dim R \geq 2$, characteristic $p > 0$, and perfect residue field. Let x_1, \dots, x_d be part of a system of parameters with $x_1, x_2 \in \tau$, where τ is the test ideal of R . Let I be an ideal and suppose $J = (x_1^t, x_2^t, x_3, \dots, x_d) \subseteq I \subseteq J^*$, where $t \geq 2$. Then $r_J(I) = 1$.

Proof. As in the proof of Theorem 3.7, we can write $I = J + (x_1 x_2)^{t-1}(u'_1, \dots, u'_s)$. Thus it suffices to see that $(x_1 x_2)^{t-1} u'_i (x_1 x_2)^{(t-1)} u'_j \in J^2$ for all $i, j \in \{1, \dots, s\}$. This is immediate, since $(x_1 x_2)^{2t-2} \in (x_1^t x_2^t) \subseteq J^2$ as long as $t \geq 2$. \square

Corollary 3.9. Let (R, \mathfrak{m}) be a Cohen–Macaulay, excellent, normal local ring with $\dim R \geq 2$, characteristic $p > 0$ and perfect residue field. Let x_1, \dots, x_d be a system of parameters with $x_1, x_2 \in \tau$, where τ is the test ideal of R . Let I be an ideal and suppose $(x_1^t, x_2^t, x_3, \dots, x_d) \subseteq I \subseteq (x_1^t, x_2^t, x_3, \dots, x_d)^*$, where $t \geq 2$. Then $\text{gr}_I R$ is Cohen–Macaulay.

Proof. Notice that with the above conditions I is an \mathfrak{m} -primary ideal and satisfies G_ℓ , where ℓ is the analytic spread. Also, the condition $\text{depth } R/I^j \geq d-g-j+1$ for all $1 \leq j \leq \ell-g+1$, where $g = \text{ht}(I)$, is immediately satisfied and by Observation 3.8 $r(I) = 1$. Therefore, by [JU, Theorem 3.1] we have $\text{gr}_I(R)$ is Cohen–Macaulay. \square

Theorem 3.10. *Let (R, \mathfrak{m}) be a normal local ring of characteristic $p > 0$ and perfect residue field. Let I be an ideal and suppose that the test ideal is \mathfrak{m} . Then $*\text{-core}(I) = (J : I)J = (J : I)I$ for any minimal $*\text{-reduction } J$ of I .*

Proof. Since $\tau \subseteq (J : I)$, we have $(J : I) = \mathfrak{m}$ or $(J : I) = R$. Note that $(J : I) = R$ is equivalent to $I = J$, which means that I is the only minimal $*\text{-reduction}$ of I , and thus the conclusion holds in this case.

Hence, we may assume that $(J : I) = \mathfrak{m}$. We have $(J : I)J \subseteq *\text{-core}(I)$ by Observation 3.1. In order to verify the other inclusion, we will check that the assumptions in part (a) of Theorem 3.4 hold. The fact that

$$(f_1, \dots, \hat{f}_i, \dots, f_n) : f_i \subseteq \mathfrak{m}$$

is immediate, since f_1, \dots, f_n is a minimal system of generators of J . It remains to check that $u(J : I) \subseteq \mathfrak{m}J$, for $u \in J^{*sp}$.

Since $u \in J^{*sp} \cap I$, then $u(J : I) \subseteq J^{*sp}$. On the other hand, $u(J : I) \subseteq J$ and thus $u(J : I) \subseteq J^{*sp} \cap J = \mathfrak{m}J$, where the last equality holds by [Vr1, Proposition 4.2]. \square

Lemma 3.11. *Let (R, \mathfrak{m}) be an excellent, analytically irreducible local ring of characteristic $p > 0$ and \mathfrak{m} -primary test ideal τ . Let f_1, \dots, f_n be $*\text{-independent}$ elements. Let $J = (f_1, \dots, f_n)$ and consider $I = (f_1, \dots, f_n, u_1, \dots, u_s)$, where $u_1, \dots, u_s \in J^{*sp}$. Then there exists q_0 sufficiently large such that for all $q \geq q_0$ the following hold:*

- (a) $(f_1^q, \dots, \hat{f}_i^q, \dots, f_n^q) : f_i^q \subseteq \tau$ for all $i = 1, \dots, n$, and
- (b) $u_k^q(J^{[q]} : u_k^q) \subseteq \mathfrak{m}\tau J^{[q]}$ for all $k = 1, \dots, s$.

Proof. Part (a) follows from Proposition 2.2.

It remains to prove part (b). Fix j sufficiently large so that $(\tau^j : \tau) \subseteq \mathfrak{m}\tau$. This is possible by the Artin–Rees Lemma, since $(\tau^j : \tau) \subseteq \tau^j : d$ for a fixed nonzerodivisor $d \in \tau$, and $Fd \in \tau^j$ implies that $Fd \in (d) \cap \tau^j \subseteq d\tau^{j-h}$, i.e. $F \in \tau^{j-h}$, where h is a constant. Since τ is \mathfrak{m} -primary, we can use Proposition 2.2 to find a q_0 so that $(f_1^q, \dots, \hat{f}_i^q, \dots, f_n^q) : f_i^q \subseteq \tau^j$ for all $q \geq q_0$ and for all $i = 1, \dots, n$.

Since $u_k \in J^{*sp}$, we can choose $q \gg 0$ so that $u_k^q \in (\tau^j J^{[q]})^* \subseteq (\tau^j J^{[q]}) : \tau$. Let $g \in (J^{[q]} : u_k^q)$ and write $gu_k^q = c_1 f_1^q + \dots + c_n f_n^q$. Multiplying by $t \in \tau$ yields $tgu_k^q = (tc_1)f_1^q + \dots + (tc_n)f_n^q$. But on the other hand we have $tu_k^q = d_1 f_1^q + \dots + d_n f_n^q$, with $d_1, \dots, d_n \in \tau^j$. It follows that $tc_i - gd_i \in (f_1^q, \dots, \hat{f}_i^q, \dots, f_n^q) : f_i^q \subseteq \tau^j$, and therefore $c_i \in (\tau^j : \tau) \subseteq \mathfrak{m}\tau$. \square

Theorem 3.12. *Let (R, \mathfrak{m}) be a normal local ring of characteristic $p > 0$, with perfect residue field and \mathfrak{m} -primary test ideal τ . Let I be an ideal and let $J \subseteq I$ be a minimal $*\text{-reduction}$ of I . Then $*\text{-core}(I^{[q]}) = (J^{[q]} : I^{[q]})J^{[q]}$ for all $q = p^e \gg 0$.*

Proof. Let $I = J + (u_1, \dots, u_s)$, where $u_j \in J^{*sp}$ for all $j = 1, \dots, s$. We need to prove $*\text{-core}(I^{[q]}) = (J^{[q]} : I^{[q]})J^{[q]}$ by showing the hypotheses of Theorem 3.4 hold when I, J are replaced by $I^{[q]}, J^{[q]}$ respectively.

First note that by Lemma 3.11 part (a), $(f_1^q, \dots, \hat{f}_i^q, \dots, f_n^q) : f_i^q \subseteq \tau \subseteq (J^{[q]} : I^{[q]})$ showing that the first hypothesis of part (a) in Theorem 3.4 is satisfied. Using Lemma 3.11 part (b) we have that $u_j^q(J^{[q]} : u_j^q) \subseteq \mathfrak{m}\tau J^{[q]} \subseteq (J^{[q]} : I^{[q]})J^{[q]}$ for all $j = 1, \dots, s$, which shows that the hypotheses of Theorem 3.4 part(a) hold.

In order to verify that the hypothesis of Theorem 3.4 part (b) holds we note that $\tau \subseteq (J^{[q]} : I^{[q]})$. Thus the result follows by Lemma 3.11 part (b). \square

4. EXAMPLES

The first example was an inspiration for Theorem 3.7 although the ring is not normal.

Example 4.1. Let $R = k[[x, y, z]]/(xyz)$, where k is a field of characteristic $p > 0$ and $I = (x, yz)$. By [Va, Theorem 3.7], $\tau = (xy, xz, yz)$ and $I = (x + ayz)^*$ for any $0 \neq a \in k$. Let $J = (x + ayz)$, for some $0 \neq a \in k$. Note that

$$((x + ayz) : I)J = (x^2, y^2z^2) = \bigcap_{a \in k} (x + ayz) = *\text{-core}(I).$$

If the parameter ideal is not sufficiently embedded for Theorem 3.7 to apply, the following example shows that $*\text{-core}(I) \neq (J : I)J$.

Example 4.2. Let $R = k[x, y, z]/(x^5 + y^5 + z^5)$, where k is a perfect field of characteristic $p > 0$ such that $p \neq 5$. Let $J = (y, z)$ and set $u = x^2$ and $I = J + (u)$. Note that $u \in J^*$ as can be seen from the fact that $\tau = \mathfrak{m}^3$ and $u \in (J : \mathfrak{m}^3)$. The assumptions in both part (a) and part (b) of Theorem 3.4 fail, and $(J : I)J \not\subseteq J' = (y + x^2, z)$, which is a minimal *-reduction of I .

We are going to use the following result of Brenner to give another example where the equality $*\text{-core}(I) = (J : I)J$ is not always obtained when the assumptions of Theorem 3.4 do not hold.

Theorem 4.3 ([Br, Example 5.2]). *If k is an algebraically closed field,*

$$R = \frac{k[x, y, z]}{(x^d + y^d + z^d)},$$

where d is not divisible by the characteristic of k , and $f = x^{d_1}, g = y^{d_2}, h = z^{d_3}$ are such that $d_i \leq d$ for $i \in \{1, 2, 3\}$, and $d_1 + d_2 + d_3 \leq 2d$, then $R_{\geq d} \subseteq (f, g, h)^$.*

Note that if R is a graded ring, J is a homogeneous ideal, and $u \in J^*$ has degree larger than the degrees of the generators of J , then $u \in J^{*sp}$.

Example 4.4. Let $R = k[x, y, z]/(x^{10} + y^{10} + z^{10})$, where k is an algebraically closed field of characteristic $p > 0$ such that $p \neq 2, 5$. Let $J = (x^5, y^7, z^8)$ and $u = xy^3z^6$. By Theorem 4.3, u is in the tight closure of J . Define $I = J + (u)$, and note

that J is a minimal $*$ -reduction of I . The assumption that $u(J : I) \subseteq (J : I)J$ fails in Theorem 3.4 because $(J : I) = (x^4, y^4, z^2)$ and $z^2u \notin (J : I)J$. We have that $(J : I)J \not\subseteq *$ -core(I). More specifically, $x^5z^2 \in (J : I)J$, but it is not in $J' = (x^5 + xu, y^7 + yu, z^8 + zu)$, which is a minimal $*$ -reduction of I .

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