

LOCAL COHOMOLOGY MODULES WITH INFINITE DIMENSIONAL SOCLES

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ABSTRACT. In this paper we prove the following generalization of a result of Hartshorne: Let T be a commutative Noetherian local ring of dimension at least two, $R = T[x_1, \dots, x_n]$, and $I = (x_1, \dots, x_n)$. Let f be a homogeneous element of R such that the coefficients of f form a system of parameters for T . Then the socle of $H_I^n(R/fR)$ is infinite dimensional.

1. INTRODUCTION

The third of Huneke's four problems in local cohomology [Hu] is to determine when $H_I^i(M)$ is Artinian for a given ideal I of a commutative Noetherian local ring R and finitely generated R -module M . An R -module N is Artinian if and only if $\text{Supp}_R N \subseteq \{m\}$ and $\text{Hom}_R(R/m, N)$ is finitely generated, where m is the maximal ideal of R . Thus, Huneke's problem may be separated into two subproblems:

- When is $\text{Supp}_R H_I^i(M) \subseteq \{m\}$?
- When is $\text{Hom}_R(R/m, H_I^i(M))$ finitely generated?

This article is concerned with the second question. For an R -module N , one may identify $\text{Hom}_R(R/m, N)$ with the submodule $\{x \in N \mid mx = 0\}$, which is an R/m -vector space called the *socle* of N (denoted $\text{soc}_R N$). It is known that if R is an unramified regular local ring, then the local cohomology modules $H_I^i(R)$ have finite dimensional socles for all $i \geq 0$ and all ideals I of R ([HS], [L1], [L2]). The first example of a local cohomology module with an infinite dimensional socle was given in 1970 by Hartshorne [Ha]: Let k be a field, $R = k[[u, v]][x, y]$, $P = (u, v, x, y)R$, $I = (x, y)R$, and $f = ux + vy$. Then $\text{soc}_{R_P} H_{I_{R_P}}^2(R_P/fR_P)$ is infinite dimensional. Of course, since I and f are homogeneous, this is equivalent to saying that $\text{Hom}_R(R/P, H_I^2(R/fR))$ (the **socle* of $H_I^2(R/fR)$) is infinite dimensional. Hartshorne proved this by exhibiting an infinite set of linearly independent elements in the **socle* of $H_I^2(R)$.

In the last 30 years there have been few results in the literature which explain or generalize Hartshorne's example. For affine semigroup rings, a remarkable result proved by Helm and Miller [HM] gives necessary and sufficient conditions (on the semigroup) for the ring to possess a local cohomology module (of a finitely generated

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module) having infinite dimensional socle. Beyond that work, however, little has been done.

In this paper we prove the following:

Theorem 1.1. *Let (T, m) be a Noetherian local ring of dimension at least two. Let $R = T[x_1, \dots, x_n]$ be a polynomial ring in n variables over T , $I = (x_1, \dots, x_n)$, and $f \in R$ a homogeneous polynomial whose coefficients form a system of parameters for T . Then the $*$ socle of $H_I^n(R/fR)$ is infinite dimensional.*

Hartshorne's example is obtained by letting $T = k[[u, v]]$, $n = 2$, and $f = ux + vy$ (homogeneous of degree 1). Note, however, that we do not require the coefficient ring to be regular, or even Cohen-Macaulay. As a further illustration, consider the following:

Example 1.2. Let $R = k[[u^4, u^3v, uv^3, v^4]][x, y, z]$, $I = (x, y, z)R$, and $f = u^4x^2 + v^8yz$. Then the $*$ socle of $H_I^3(R/fR)$ is infinite dimensional.

Part of the proof of Theorem 1.1 was inspired by the recent work of Katzman [Ka] where information on the graded pieces of $H_I^n(R/fR)$ is obtained by examining matrices of a particular form. We apply this technique in the proof of Lemma 2.8.

Throughout, all rings are assumed to be commutative with identity. The reader should consult [Mat] or [BH] for any unexplained terms or notation and [BS] for the basic properties of local cohomology.

2. THE MAIN RESULT

Let $R = \bigoplus R_\ell$ be a Noetherian ring graded by the nonnegative integers. Assume R_0 is local and let P be the homogeneous maximal ideal of R . Given a finitely generated graded R -module M we define the $*$ socle of M by

$$\begin{aligned} * \operatorname{soc}_R M &= \{x \in M \mid Px = 0\} \\ &\cong \operatorname{Hom}_R(R/P, M). \end{aligned}$$

Clearly, $* \operatorname{soc}_R M \cong \operatorname{soc}_{R_P} M_P$. An interesting special case of Huneke's third problem is the following:

Question 2.1. *Let $n := \mu_R(R_+/PR_+)$, the minimal number of generators of R_+ . When is $* \operatorname{soc} H_{R_+}^n(R)$ finitely generated?*

For $i \in \mathbb{N}$ it is well known that $H_{R_+}^i(R)$ is a graded R -module, each graded piece $H_{R_+}^i(R)_\ell$ is a finitely generated R_0 -module, and $H_{R_+}^i(R)_\ell = 0$ for all sufficiently large integers ℓ ([BS, 15.1.5]). If we know *a priori* that $H_{R_+}^n(R)_\ell$ has finite length for all ℓ (e.g., if $\operatorname{Supp}_R H_{R_+}^n(R) \subseteq \{P\}$), then Question 2.1 is equivalent to:

Question 2.2. *When is $\operatorname{Hom}_R(R/R_+, H_{R_+}^n(R))$ finitely generated?*

We give a partial answer to these questions for hypersurfaces. For the remainder of this section we adopt the following notation: Let (T, m) be a local ring of dimension d and $R = T[x_1, \dots, x_n]$ a polynomial ring in n variables over T . We endow R with an \mathbb{N} -grading by setting $\deg T = 0$ and $\deg x_i = 1$ for all i . Let $I = R_+ = (x_1, \dots, x_n)R$ and let $P = m + I$ be the homogeneous maximal ideal of R . Let $f \in R$ be a homogeneous element of degree p and C_f the ideal of T generated by the coefficients of f .

Our main result is the following:

Theorem 2.3. *Assume $d \geq 2$ and the (nonzero) coefficients of f form a system of parameters for T . Then ${}^* \text{soc}_R H_T^n(R/fR)$ is not finitely generated.*

The proof of this theorem will be given in a series of lemmas below. Before proceeding with the proof we make a couple of remarks:

- Remark 2.4.* (a) If $d \leq 1$ in Theorem 2.3, then ${}^* \text{soc} H_T^n(R/fR)$ is finitely generated. This follows from [DM, Corollary 2] since $\dim R/I = \dim T \leq 1$.
 (b) The hypothesis that the nonzero coefficients of f form a system of parameters for T is stronger than our proof requires. One only needs that C_f be m -primary and that there exists a dimension 2 ideal containing all but two of the coefficients of f . (See the proof of Lemma 2.8.)

The following lemma identifies the support of $H_T^n(R/fR)$ for a homogeneous element $f \in R$. This lemma also follows from a much more general result recently proved by Katzman and Sharp [KS, Theorem 1.5].

Lemma 2.5. *Let $f \in R$ be a homogeneous element. Then*

$$\text{Supp}_R H_T^n(R/fR) = \{Q \in \text{Spec } R \mid Q \supseteq I + C_f\}.$$

Proof. It is enough to prove that $H_T^n(R/fR) = 0$ if and only if $C_f = T$. As $H_T^n(R/fR)_k$ is a finitely generated T -module for all k , we have by Nakayama that $H_T^n(R/fR) = 0$ if and only if $H_T^n(R/fR) \otimes_T T/m = 0$. Now

$$\begin{aligned} H_T^n(R/fR) \otimes_T T/m &\cong H_T^n(R/fR \otimes_T T/m) \\ &\cong H_N^n(S/fS) \end{aligned}$$

where $S = (T/m)[x_1, \dots, x_n]$ is a polynomial ring in n variables over a field and $N = (x_1, \dots, x_n)S$. As $\dim S = n$, we see that $H_N^n(S/fS) = 0$ if and only if the image of f modulo m is nonzero. Hence, $H_T^n(R/fR) = 0$ if and only if at least one coefficient of f is a unit, i.e., $C_f = T$. \square

We are mainly interested in the case when the coefficients of f generate an m -primary ideal:

Corollary 2.6. *Let $f \in R$ be homogeneous and suppose C_f is m -primary. Then*

$$\text{Supp}_R H_T^n(R/fR) = \{P\}.$$

Our next lemma is the key technical result in the proof of Theorem 2.3.

Lemma 2.7. *Suppose $u, v \in T$ such that $\text{ht}(u, v)T = 2$. For each integer $n \geq 1$ let M_n be the cokernel of $\phi_n : T^{n+1} \rightarrow T^n$ where ϕ_n is represented by the matrix*

$$A_n = \begin{pmatrix} u & v & 0 & 0 & \cdots & 0 & 0 \\ 0 & u & v & 0 & \cdots & 0 & 0 \\ 0 & 0 & u & v & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & u & v \end{pmatrix}_{n \times (n+1)}.$$

Let $J = \bigcap_{n \geq 1} \text{ann}_T M_n$. Then $\dim T/J = \dim T$.

Proof. Let \hat{T} denote the m -adic completion of T . Then $\text{ht}(u, v)\hat{T} = 2$, $\text{ann}_T M_n = \text{ann}_{\hat{T}}(M_n \otimes_T \hat{T}) \cap T$, and $\dim T/(I \cap T) \geq \dim \hat{T}/I$ for all ideals I of \hat{T} . Thus, we may assume T is complete. Now let p be a prime ideal of T such that $\dim T/p = \dim T$. Since T is catenary, $\text{ht}(u, v)T/p = 2$. Assume the lemma is true for complete domains. Then $\bigcap_{n \geq 1} \text{ann}_{T/p}(M_n \otimes_T T/p) = p/p$. Hence

$$\begin{aligned} J &= \bigcap_{n \geq 1} \text{ann}_T M_n \\ &\subseteq \bigcap_{n \geq 1} \text{ann}_T(M_n \otimes_T T/p) \\ &= p, \end{aligned}$$

which implies that $\dim T/J \geq \dim T/p = \dim T$. Thus, it suffices to prove the lemma for complete domains.

As T is complete, the integral closure S of T is a finite T -module ([Mat, page 263]). Since $\text{ht}(u, v)S = 2$ ([Mat, Theorem 15.6]) and S is normal, $\{u, v\}$ is a regular sequence on S . It is easily seen that $I_n(A_n)$, the ideal of $n \times n$ minors of A_n , is $(u, v)^n T$. By the main result of [BE] we obtain $\text{ann}_S(M_n \otimes_T S) = (u, v)^n S$. Hence $\text{ann}_T M_n \subseteq (u, v)^n S \cap T$. As S is a finite T -module, there exists an integer k such that $\text{ann}_T M_n \subseteq (u, v)^{n-k} T$ for all $n \geq k$. Therefore, $\bigcap_{n \geq 1} \text{ann}_T M_n = (0)$, which completes the proof. \square

Lemma 2.8. *Assume $d \geq 2$ and let $f \in R$ be a homogeneous element of degree p such that the coefficients of f form a system of parameters for T . Then $\dim T/\text{ann}_T H_1^n(R/fR) \geq 2$.*

Proof. Let c_1, \dots, c_d be the nonzero coefficients of f . Let $T' = T/(c_3, \dots, c_d)T$ and $R' = T'[x_1, \dots, x_n] \cong R/(c_3, \dots, c_d)R \cong R \otimes_T T'$. Since

$$\begin{aligned} \dim T/\text{ann}_T H_1^n(R/fR) &\geq \dim T/\text{ann}_T(H_1^n(R/fR) \otimes_T T') \\ &= \dim T'/\text{ann}_{T'} H_{1R'}^n(R'/fR'), \end{aligned}$$

we may assume that $\dim T = 2$ and f has exactly two nonzero terms.

For any $w \in R$ there is a surjective map $H_1^n(R/wfR) \rightarrow H_1^n(R/fR)$. Hence, $\text{ann}_T H_1^n(R/wfR) \subseteq \text{ann}_T H_1^n(R/fR)$. Thus, we may assume that the terms of f have no (non-unit) common factor. Without loss of generality, we may write $R = T[x_1, \dots, x_k, y_1, \dots, y_r]$ and $f = ux_1^{d_1} \dots x_k^{d_k} + vy_1^{e_1} \dots y_r^{e_r} = u\mathbf{x}^{\mathbf{d}} + v\mathbf{y}^{\mathbf{e}}$, where $\{u, v\}$ is a system of parameters for T . As f is homogeneous, $p = \sum_i d_i = \sum_i e_i$.

Applying the right exact functor $H_1^n(\cdot)$ to $R(-p) \xrightarrow{f} R \rightarrow R/fR \rightarrow 0$ we obtain the exact sequence

$$H_1^n(R)_{-\ell-p} \xrightarrow{f} H_1^n(R)_{-\ell} \rightarrow H_1^n(R/fR)_{-\ell} \rightarrow 0$$

for each $\ell \in \mathbb{Z}$. For each ℓ , $H_1^n(R)_{-\ell}$ is a free T -module with basis

$$\{\mathbf{x}^{-\alpha}\mathbf{y}^{-\beta} \mid \sum_{i,j} \alpha_i + \beta_j = \ell, \alpha_i > 0, \beta_j > 0 \forall i, j\}$$

(e.g., [BS, Example 12.4.1]). Let q be an arbitrary positive integer and let $\ell(q) = qp + k + r$. Define $L_{-\ell(q)}$ to be the free T -summand of $H_1^n(R)_{-\ell(q)}$ spanned by the set

$$\{\mathbf{x}^{-s\mathbf{d}-1}\mathbf{y}^{-t\mathbf{e}-1} \mid s + t = q, s, t \geq 0\}.$$

Then the cokernel of $\delta_q : L_{-\ell(q+1)} \xrightarrow{f} L_{-\ell(q)}$ is a direct summand (as a T -module) of $H_I^n(R/fR)_{-\ell(q)}$. For a given q we order the basis elements for $L_{-\ell(q)}$ as follows:

$$x^{-sd-1}\mathbf{y}^{-te-1} > x^{-s'd-1}\mathbf{y}^{-t'e-1}$$

if and only if $s > s'$. With respect to these ordered bases, the matrix representing δ_q is

$$\begin{pmatrix} u & v & 0 & 0 & \cdots & 0 & 0 \\ 0 & u & v & 0 & \cdots & 0 & 0 \\ 0 & 0 & u & v & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & u & v \end{pmatrix}_{(q+1) \times (q+2)}$$

By Lemma 2.7, if $J = \bigcap_{q \geq 1} \text{ann}_T \text{coker } \delta_q$, then $\dim T/J = \dim T = 2$. As $\text{coker } \delta_q$ is a direct T -summand of $H_I^n(R/fR)$, we have $\text{ann}_T H_I^n(R/fR) \subseteq J$. This completes the proof. □

Lemma 2.9. *Under the assumptions of Lemma 2.8, $\text{Hom}_R(R/I, H_I^n(R/fR))$ is not finitely generated as an R -module. Consequently, $\text{Hom}_R(R/I, H_I^n(R/fR))_k \neq 0$ for infinitely many k .*

Proof. Suppose $\text{Hom}_R(R/I, H_I^n(R/fR))$ is finitely generated. By Lemma 3.5 of [MV] we have that $I + \text{ann}_R H_I^n(R/fR)$ is P -primary. (One should note that the hypothesis in [MV, Lemma 3.5] that the ring be complete is not necessary.) This implies that $\text{ann}_R H_I^n(R/fR) \cap T = \text{ann}_T H_I^n(R/fR)$ is m -primary, contradicting Lemma 2.8. □

We now give the proof of our main result:

Proof of Theorem 2.3. By Corollary 2.6, $\text{Supp}_R H_I^n(R/fR) = \{P\}$. Thus, $\text{Hom}_R(R/I, H_I^n(R/fR))_k$ has finite length as a T -module for all k and is nonzero for infinitely many k by Lemma 2.9. Consequently,

$$\text{Hom}_R(R/P, H_I^n(R/fR))_k = \text{Hom}_T(T/m, \text{Hom}_R(R/I, H_I^n(R/fR))_k)$$

is nonzero for infinitely many k . Hence

$$* \text{soc}_R(H_I^n(R/fR)) = \text{Hom}_R(R/P, H_I^n(R/fR))$$

is not finitely generated. □

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