STAT 345 Fall 2018
Homework 2 - Probability Name: ________________________________

Please adhere to the homework rules as given in the Syllabus.

1. **Birthday Paradox.** Suppose n people are at a party.

a) What is the probability that at least two people share a birthday? *Hint: It is easier to first look at the complement of this event.* Using a computer, create a plot showing this probability as a function of n.

b) How many people need to be in the room before this probability is greater than 1/2? Is this surprising?

c) *Pigeonhole Principle.* How many people need to be in the room before this probability is 1? Explain the intuition behind this.

2. Probability Rules. Fred is about to start watching "Game of Thrones" and "Westworld". The probability that Fred likes "Game of Thrones" is 0.6. The probability that Fred likes "Westworld" is 0.5. The probability that Fred likes both shows is 0.4. Define the events

$$A = \{\text{Fred likes Game of Thrones}\} \qquad B = \{\text{Fred likes Westworld}\}$$

a) Are A and B independent? Are they disjoint? Justify your answers mathematically.

b) Find the probability that Fred doesn't like Game of Thrones or Westworld.

c) Find the probability that Fred likes Westworld, given that he doesn't like Game of Thrones.

3. Probability of Error. Professor Halfbrain has to grade n exams. Let E_i be the probability that he makes a mistake grading the i^{th} exam, and suppose that $P(E_i) = \alpha$. Assuming that the events are independent, find the probability that he makes a mistake on at least one exam. Using a computer, plot this probability as a function of n (assuming that $\alpha = 0.05$). How many students can the Professor handle before the probability of making a mistake is more than 1/2?

4. Conditional Probability. Suppose we roll a fair six sided die one time and define the following events: $A = \{\text{We roll a three}\}$, $B = \{\text{We roll an odd number}\}$, $C = \{\text{We do not roll a 1}\}$. Use the definition of conditional probability.

a) Determine $P(A|B)$.

b) Determine $P(A|C)$.

c) Determine $P(A|D)$ where $D = B \cap C$.

5. Schrodingers Cat. *I apologize in advance for this morbid problem.* Schrodinger asks his neighbor to feed his elderly cat while he is gone on vacation. If the neighbor remembers, the cat will die with probability 0.1. If the neighbor forgets, the cat will die with probability 0.8. Schrodinger is 90% sure that the neighbor will remember to feed the cat.

a) What is the probability that Schrodingers cat will be alive when he returns from vacation (ignoring quantum mechanics).

b) If the cat is dead when Schrodinger returns, what is the probability that his neighbor forgot to feed it?

6. **Jar of Coins.** *Note: This is adapted from a Google interview question.* I have a jar with 100 quarters. Most of them are regular (fair) quarters, but 12 of them are fake! Of these fake quarters, 7 have heads on both sides and 5 have tails on both sides.

a) If you select a coin from the jar at random and flip it 3 times and get heads every time, what is the probability that this is a "two-heads" quarter?

b) Repeat the calculation in part a) but this time you flip the coin k times, getting heads each time. What is the probability that this is a "two-heads" quarter (in terms of k)?

c) Using a computer, plot this probability as a function of k. For what value of k is the probability more than 1/2? For what value of k is the probability more than 0.9?

7. Challenge Problem. *The Dating Problem.* Alf is ready to find a wife and settle down, so naturally, he turns to Tinder. Alf has N matches, each of which can be ranked from 1 to N. His goal is to find the best possible match with high probability. He decides to use the following probabilistic dating scheme.

- He will date each of his matches sequentially (in a completely random order). At the end of the date, he will make an *immediate* and *irrevocable* decision on whether or not to propose. (Yeah, Alf moves fast).
- For some integer $x > 1$, he will automatically reject the first $x - 1$ matches. He will then propose to the i^{th} match only if i^{th} date was better than all of the previous $i - 1$ dates.

a) Let $x = 2$ and find the probability that Alf finds his soulmate, i.e. propses to the best possible match. *Hint: Let A be the event that Alf proposes to the best match, and let B_i be the event that the i^{th} date is the best one. Use Law of Total Probability.*

b) Generalize your answer for $x < N$.

c) When N is large, the following identity holds (approximately).

$$\frac{x-1}{N}\sum_{i=x}^{N}\frac{1}{i-1} \approx -\frac{x}{N}\ln(x/N)$$

Show (using calculus) that the choice $x = N/e$ maximizes the probability of hiring the best candidate. If Alf has $N = 100$ matches, how many dates should he go on before he starts thinking about proposing? What is the probability that he finds his soulmate?