STAT 345 Spring 2018
Homework 5 - Continuous Random Variables   Name: ____________________

Please adhere to the homework rules as given in the Syllabus.

1. **Trapezoidal Distribution.** Consider the following probability density function.

$$f(x) = \begin{cases} \frac{x+1}{5}, & -1 \leq x < 0 \\ \frac{1}{5}, & 0 \leq x < 4 \\ \frac{5-x}{5}, & 4 \leq x \leq 5 \\ 0, & otherwise \end{cases}$$

a) Sketch the PDF.

b) Show that the area under the curve is equal to 1 using geometry.

c) Show that the area under the curve is equal to 1 by integrating. (You will have to split the integral into peices).

d) Find $P(X < 3)$ using geometry.

e) Find $P(X < 3)$ with integration.

2. **PDF to CDF**. For each of the following PDFs, find and sketch the CDF. The Median (M) of a continuous distribution is defined by the property $F(M) = 0.5$. Use the CDF to find the Median of each distribution.

a) (Special case of Beta Distribution)

$$f(x) = 1.5\sqrt{x}, \quad 0 < x < 1$$

b) (Special case of Pareto Distribution)

$$f(x) = \frac{1}{x^2}, \quad x > 1$$

3. **CDF to PDF**. For each of the following CDFs, find the PDF. Use the PDF you derived to compute the expected value of each distribution.

a)

$$F(x) = \begin{cases} 0, & x < 0 \\ 1 - (1 - x^2)^2, & 0 \le x < 1 \\ 1, & x \ge 1 \end{cases}$$

b) *Don't worry about finding the expected value of this guy. We will learn a really interesting trick for finding integrals of this form, but we haven't gotten to it yet..*

$$F(x) = \begin{cases} 0, & x < 0 \\ 1 - e^{-x^2}, & x \ge 0 \end{cases}$$

4. Waiting Times. Suppose that the time it takes for Jules to get his food at Big Kahuna Burger is a random variable X with the following CDF (*Note: This is an Exponential distribution with an expected waiting time of 1 minute.*).

$$F_X(x) = 1 - e^{-x}, \ x > 0$$

a) Find the probability that Jules waits at least 1 minute before getting his tasty burger.

b) Given that Jules has already waited 5 minutes, what is the probability he must wait at least 1 more minute before getting his tasty burger. Compare this answer to part a). *Hint: Use the definition of Conditional Probability to find* $P(X > 6|X > 5)$.

c) Jules is happy as long as he gets his burger within 2 minutes. Let $Y = 1$ if Jules is happy and $Y = 0$ otherwise. What is the distribution of Y? Also give the mean and variance of Y.

d) Jules visits Big Kahuna Burger every monday for a year. Each time he records his waiting time X_i and the corresponding random variable Y_i defined above for $i = 1, 2, \cdots 52$. Assume each visit is independent, and let $Z = \sum_{i=1}^{52} Y_i$. What is the distribution, mean and variance of Z?

5. Project Dank. Professor Halfbrain is doing "extensive research" on the beer Project Dank from La Cumbre Brewing. The Professor has determined that, if X is the number of IBU's in a batch of Project Dank, then the probability density function of X is

$$f(x) = \begin{cases} \frac{c}{x^{10}}, & x \geq 100 \\ 0, & otherwise \end{cases}$$

a) Find the value of c which makes $f(x)$ a valid PDF. Sketch the PDF.

b) Find the mean and standard deviation of X.

c) Find the probability that a batch of Project Dank has more than 140 IBU's.

d) Assuming each batch is independent, what is the probability that exactly 2 out of 5 batches have more than 140 IBU's.

6. Expected Values.

a) If X is a Continuous Uniform RV with $a = 0$ and $b = \pi$, then the density function is given by $f(x) = 1/\pi$ for $0 < x < \pi$. Find $E(\pi^2 \sin(X))$.

b) Let X have PDF, $f(x) = 2x$ for $0 < x < 1$. Use integration by parts or tabular integration to find $E(Xe^X)$.

c) If X is an Exponential RV with rate λ, then the density function is given by $f(x) = \lambda e^{-\lambda x}$ for $x > 0$. Find $E(a^X)$ where $a > 0$. Does $E(a^X)$ exist for all values of $a > 0$? If not, give a condition on a for which the expected value exists. *Hint:* $a^x e^{-\lambda x} = (ae^{-\lambda})^x$

7. Challenge Problem: The Moment Genrating Function (MGF) of a random variable X is defined as

$$M_X(t) = E(e^{Xt})$$

Inside the expected value, t can be regarded as a constant. However, $M_X()$ is a function of this t. (*Note: If you are familiar with Laplace Transforms, note that for continuous* X*, the MGF is essentially the Laplace Transform of the PDF*).

a) Find the Moment Generating Function of $X \sim Exp(\lambda)$. You will need to look the PDF $f(x)$ up in Table 1. *Hint: With some manipulation, you can use the result from problem* 6c.

Moment Generating Functions are useful for a variety of reasons (the most important of which will appear as a future challenge problem). As the name suggests, they can be useful for finding moments of the distribution. Specifically, the following identity holds for any non-negative integer k.

$$E(X^k) = M^{(k)}(0)$$

Notation: $M_X^{(k)}(0)$ *is the* k^{th} *derivative of* $M_X(t)$ *with respect to* t*, evaluated at* 0.

b) Use this identity to find $E(X)$ and $E(X^2)$ for $X \sim Exp(\lambda)$. You should get the same answer as with the usual method.

I should note that the MGF does not exist for some distributions, in which case this method cannot be used. Now let's try to see why the MGF works!

c) Expand $M_X(t) = E(e^{Xt})$ by writing e^{Xt} as a Taylor Series expansion. Now use linearity of expectation. Explain (no need for formal proof) why the identity holds.