STAT 345 Fall 2018
Homework 7 - Multiple Random Variables   Name: ______________________

Please adhere to the homework rules as given in the Syllabus.

1. Consider the following joint mass function for discrete random variables X and Y.

$$f_{X,Y}(x,y) = \begin{cases} \frac{cx}{2^y}, & y = 1,2,3 \text{ and } x = 1, \cdots y \\ 0, & otherwise \end{cases}$$

The idea is that x and y can both take values 1, 2 and 3 but x must be less than or equal to y.

a) Find the value of c which makes this a valid joint mass function. *Hint: There are 6 possible combinations of X and Y. Write them out and evaluate $f(x,y)$ for each case.*

b) What is the probability that $X + Y \geq 5$?

c) Find $E(XY)$

2. Chili Peppers. The *Rate my Professor* website recently eliminated the *Chili Pepper* feature on the website. Intuitively, whether or not a professor has a Chili Pepper should have nothing to do with their effectiveness as a teacher, but a 2006 paper by Rionolo et al. showed a link between professor evaluations and perceived attractiveness. Suppose that X is a random variable describing the *Quality* ranking that a student gives to a professor (5 being highest quality), and Y is a random variable which is 1 is the student gives a Chili Pepper and 0 otherwise. The following table describes the Joint PMF of X and Y (based on a sample of 200 professor ratings, simulated via data from Rionolo's paper).

	x=1	x=2	x=3	x=4	x=5
y=0	0.01	0.05	0.15	0.2	0.1
y=1	0.005	0.02	0.06	0.16	0.245

a) Explain why this is a valid Joint PMF.

b) Find $P(Y = 1, X \geq 3)$, and $P(Y = 0, X \geq 3)$

c) Find the Marginal PMFs of X and Y.

d) Are X and Y independent? Why or why not?

f) Given that Professor Halfbrain has a chili pepper on Rate my Professor, what is the average rating a student will give him? What is the probability that a student gives him at least a 3? *Hint: Find the conditional distribution of X given that $Y = 1$.*

3. Correlation. Let $X \sim Exp(1)$.

a) Use the Gamma function to show that $E(X^k) = k!$ for a positive integer k. (*we did this in class*)

a) Let $Y = X^3$. Show that $Var(Y) = 684$. *Hint:* $Y^2 = (X^3)^2 = X^6$

b) Find $Cor(X, Y)$.

4. Linear Model. Let $X \sim U(0, 2)$, $\epsilon \sim N(0, 1)$ and assume X and ϵ are independent. Now suppose $Y = X + \epsilon$.

a) Find the expected value and the variance of both X and Y.

b) Find $E(XY)$. *Hint: Notice that* $E(XY) = E(X(X + \epsilon))$. *Now use linearity of expectation and independence.*

c) Use the above to find $Cov(X, Y)$ and then $Cor(X, Y)$.

5. **Linearity of Expectation.** A mail-man must deliver mail to n different mailboxes. He has n letters to deliver, exactly one letter for each box. Unfortunately, the mail-man had a "few too many" last night and places each letter into a mailbox at random. *Note: This problem is easy. Don't overthink things.*

a) Suppose you are the owner of the i^{th} mailbox. What is the probability that you get your letter?

b) Let X_i be a Bernoulli random variable, where $X_i = 1$ if the owner of the i^{th} mailbox gets his/her letter and $X_i = 0$ otherwise. What is $E(X_i)$?

c) If we let $Y = \sum_{i=1}^{n} X_i$, then Y is the total number of people who get their letter. Using Linearity of Expectation, what is the expected number of people who get their letter?

6. **Bivariate Bernoulli Distribution**. A disc is checked for scratch and shock resistance. Let $X = 1$ if the disc has high scratch resistance and $X = 0$ otherwise. Let $Y = 1$ if the disc has high shock resistance and $Y = 0$ otherwise. Marginally, the distribution of X is Bernoulli with $p = 0.7$. Additionally, assume that $f_{X,Y}(1,1) = 0.5$ and $f_{X,Y}(0,0) = 0.2$.

a) Complete the table giving the Joint Probability Mass Function for X and Y.

	$x = 0$	$x = 1$
$y = 0$		
$y = 1$		

b) Give the marginal distributions of X and Y.

c) Find the correlation of X and Y.

7. Linear Functions of Random Variables.

a) Let $X_1 \sim Exp(1)$, $X_2 \sim Exp(2)$ and $X_3 \sim Exp(3)$ and assume that they are independent of eachother. Let $Y = X_1 + 4X_2 - 6X_3$. Find the mean and variance of Y.

b) Let $U \sim Binom(20, 0.5)$, $V \sim Poiss(4)$ and $Cor(U, V) = -1/3$. Find $E(W)$ and $Var(W)$ where $W = 3U - V/2$.

c) Let X_1 and X_2 be independent random variables with mean μ and variance σ^2 and let p be a constant such that $0 < p < 1$. Find the mean and variance of $Y = pX_1 + (1 - p)X_2$. Use a calculus argument to show that $Var(Y)$ is minimized when $p = 1/2$, i.e. for $Y = \frac{X_1+X_2}{2}$.

8. General Functions of Random Variables.

a) Timothy will flip a fair coin until he gets a Tails. Lt X be the number of flips, and let X^2 be the number of points Timothy scores. Find the PMF of Y, and find $P(Y > 10)$.

b) Jimothy will flip a fair coin 5 times. Let X be the number of times he gets Tails, and let $Y = 2^X$ be the number of points that Jimothy scores. Find the PMF of Y and find $P(Y > 10)$. Is Timothy or Jimothy more likely to score more than 10 points?

9. **Challenge Problem:** Timothy throws a baseball with speed a speed of v meters per second at an angle A above the ground. The ball will land on the ground at a distance of

$$R = \frac{v^2}{g} \sin 2A$$

from where Timothy was standing (where $g = 10$ meters per second squared is the gravitational constant).

a) If $v = 10$ m/s is fixed and known, but the angle A is uniformly distributed between 0 and $\pi/2$, find the CDF and PDF of R. *Hint: Find the CDF of R first and then take the derivative.*

b) Jimothy is standing in front of Timothy. He can catch the ball if Timothy throws it between 6 and 8 meters. What is the probability that Jimothy catches the ball?

c) Now assume that the speed v is also random, and distributed as

$$v \sim Gamma(r = 1000, \lambda = 100)$$

Use Monte Carlo (notes on webpage) to approximate the mean and variance of R. Also approximate the probability that Jimothy catches the ball. Finally, provide a Histogram of the distribution of R.