STAT 345 Spring 2018
Homework 5 - Continuous Random Variables   Name: ____________________

Please adhere to the homework rules as given in the Syllabus.

1. **Trapezoidal Distribution.** Consider the following probability density function.

$$f(x) = \begin{cases} \frac{x+1}{5}, & -1 \le x < 0 \\ \frac{1}{5}, & 0 \le x < 4 \\ \frac{5-x}{5}, & 4 \le x \le 5 \\ 0, & otherwise \end{cases}$$

a) Sketch the PDF.

b) Show that the area under the curve is equal to 1 using geometry.

c) Show that the area under the curve is equal to 1 by integrating. (You will have to split the integral into peices).

d) Find $P(X < 3)$ using geometry.

e) Find $P(X < 3)$ with integration.

2. Let X have the following PDF, for $\theta > 0$.

$$f(x) = \begin{cases} \frac{c}{\theta^2}(\theta - x), & 0 < x < \theta \\ 0, & otherwise \end{cases}$$

a) Find the value of c which makes $f(x)$ a valid PDF. Sketch the PDF.

b) Find the mean and standard deviation of X.

c) Find $P\left(\frac{\theta}{10} < X < \frac{3\theta}{5}\right)$.

3. **PDF to CDF**. For each of the following PDFs, find and sketch the CDF. The Median (M) of a continuous distribution is defined by the property $F(M) = 0.5$. Use the CDF to find the Median of each distribution.

a) (Special case of Beta Distribution)

$$f(x) = 1.5\sqrt{x}, \quad 0 < x < 1$$

b) (Special case of Pareto Distribution)

$$f(x) = \frac{1}{x^2}, \quad x > 1$$

4. **CDF to PDF**. For each of the following CDFs, find the PDF.

a)

$$F(x) = \begin{cases} 0, & x < 0 \\ 1 - (1 - x^2)^2, & 0 \leq x < 1 \\ 1, & x \geq 1 \end{cases}$$

b)

$$F(x) = \begin{cases} 0, & x < 0 \\ 1 - e^{-x^2}, & x \geq 0 \end{cases}$$

5. Expected Values.

a) Continuous uniform distribution: Let X have PDF $f(x) = 1/\pi$ for $0 < x < \pi$. Find $E(\pi^2 \sin(X))$.

b) Let X have PDF, $f(x) = 2x$ for $0 < x < 1$. Use integration by parts or tabular integration to find $E(Xe^X)$.

c) Exponential distribution: Let X have PDF $f(x) = \lambda e^{-\lambda x}$ for $x > 0$ where $\lambda > 0$ is a constant. Find $E(a^X)$ where $a > 0$. Does $E(a^X)$ exist for all values of $a > 0$? If not, give a condition on a for which the expected value exists. *Hint:* $a^x e^{-\lambda x} = (ae^{-\lambda})^x$

6. Waiting Times. Suppose that the time it takes for Jules to get his food at Big Kahuna Burger is a random variable X with the following CDF (*Note: This is an Exponential distribution with an expected waiting time of 1 minute.*).

$$F_X(x) = 1 - e^{-x},\ x > 0$$

a) Find the probability that Jules waits at least 1 minute before getting his tasty burger.

b) Given that Jules has already waited 5 minutes, what is the probability he must wait at least 1 more minute before getting his tasty burger. Compare this answer to part a). *Hint: Use the definition of Conditional Probability to find* $P(X > 6|X > 5)$.

c) Jules is happy as long as he gets his burger within 2 minutes. Let $Y = 1$ if Jules is happy and $Y = 0$ otherwise. What is the distribution of Y? Also give the mean and variance of Y.

c) Jules visits Big Kahuna Burger every monday for a year. Each time he records his waiting time X_i and the corresponding random variable Y_i defined above for $i = 1, 2, \cdots 52$. Assume each visit is independent, and let $Z = \sum_{i=1}^{52} Y_i$. What is the distribution, mean and variance of Z?

7. Challenge Problem: The Moment Genrating Function (MGF) of a random variable X is defined as

$$M_X(t) = E(e^{Xt})$$

Inside the expected value, t can be regarded as a constant. However, $M_X()$ is a function of this t. (*Note: If you are familiar with Laplace Transforms, note that for continuous X, the MGF is essentially the Laplace Transform of the PDF*).

a) Find the Moment Generating Function of $X \sim Exp(\lambda)$. You will need to look the PDF $f(x)$ up in Table 1. *Hint: With some manipulation, you can use the result from problem* 5c.

Moment Generating Functions are useful for a variety of reasons (the most important of which will appear as a future challenge problem). As the name suggests, they can be useful for finding moments of the distribution. Specifically, the following identity holds for any non-negative integer k.

$$E(X^k) = M^{(k)}(0)$$

Notation: $M_X^{(k)}(0)$ *is the* k^{th} *derivative of* $M_X(t)$ *with respect to* t, *evaluated at* 0.

b) Use this identity to find $E(X)$ and $E(X^2)$ for $X \sim Exp(\lambda)$. You should get the same answer as with the usual method.

I should note that the MGF does not exist for some distributions, in which case this method cannot be used. Now let's try to see why the MGF works!

c) Expand $M_X(t) = E(e^{Xt})$ by writing e^{Xt} as a Taylor Series expansion. Now use linearity of expectation. Explain (no need for formal proof) why the identity holds.